Robert’s other metric

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Abstract

In the 1990’s Robert Bartnik found a null quasi-spherical (NQS) metric for which the Einstein equations become a particularly simple characteristic transport system coupled to a time evolution equation. Robert employed me as a postdoc/programmer, and together we turned his NQS metric into a pseudospectral code for solving the full Einstein equations for radiating black hole spacetimes.

A conference on the past and future 30 years of gravity research in Australasia seems like an appropriate time to reveal Robert’s other metric, which due to Robert’s illness, was impractical to follow up on or publish on at the time. It remains an unexplored opportunity for future research in numerical relativity.

Recap: Null Quasi-Spherical (NQS) metric

NQS coordinates and metric functions

Coordinates $\{z, r, \theta, \phi\}$.
The 3-surfaces $z = \text{const.}$ are $\approx$ forward null-cones, $\langle dz, dz\rangle = 0$.
The metric contains 6 arbitrary functions: Two scalar fields u, v and two S^2 vector fields,

$$\vec{\beta} == \beta 1\, \hat{\theta} + \beta 2\, \hat{\phi}\,, \qquad \vec{\gamma} == \gamma 1\, \hat{\theta} + \gamma 2\, \hat{\phi}$$

The null-cones are foliated by *metric* 2-spheres,

$$\{g_{ij}\} == \begin{pmatrix} g_{zz} & g_{zr} & g_{z\theta} & g_{z\phi} \\ g_{rz} & g_{rr} & g_{r\theta} & g_{r\phi} \\ g_{\theta z} & g_{\theta r} & g_{\theta\theta} & g_{\theta\phi} \\ g_{\phi z} & g_{\phi r} & g_{\phi\theta} & g_{\phi\phi} \end{pmatrix}$$

$$== \begin{pmatrix} -2\,u\,v + \gamma 1^2 + \gamma 2^2 & -u + \beta 1\,\gamma 1 + \beta 2\,\gamma 2 & r\,\gamma 1 & r\,\gamma 2\,\mathrm{Sin}[\theta] \\ -u + \beta 1\,\gamma 1 + \beta 2\,\gamma 2 & \beta 1^2 + \beta 2^2 & r\,\beta 1 & r\,\beta 2\,\mathrm{Sin}[\theta] \\ r\,\gamma 1 & r\,\beta 1 & r^2 & 0 \\ r\,\gamma 2\,\mathrm{Sin}[\theta] & r\,\beta 2\,\mathrm{Sin}[\theta] & 0 & r^2\,\mathrm{Sin}[\theta]^2 \end{pmatrix}$$

$$\sqrt{-\det[g]} == r^2\,u\,\mathrm{Sin}[\theta]$$

In general, the r-coordinate lines are *not* null-geodesics: $g_{rr} = \langle \partial_r, \partial_r \rangle = \langle \vec{\beta}, \vec{\beta} \rangle \neq 0$.
The inverse metric is,

$$\{g^{ij}\} == \begin{pmatrix} 0 & -\frac{1}{u} & \frac{\beta 1}{r\,u} & \frac{\beta 2\,\mathrm{Csc}[\theta]}{r\,u} \\ -\frac{1}{u} & \frac{2\,v}{u} & \frac{-2\,v\,\beta 1+\gamma 1}{r\,u} & \frac{(-2\,v\,\beta 2+\gamma 2)\,\mathrm{Csc}[\theta]}{r\,u} \\ \frac{\beta 1}{r\,u} & \frac{-2\,v\,\beta 1+\gamma 1}{r\,u} & \frac{u+2\,v\,\beta 1^2-2\,\beta 1\,\gamma 1}{r^2\,u} & \frac{(2\,v\,\beta 1\,\beta 2-\beta 2\,\gamma 1-\beta 1\,\gamma 2)\,\mathrm{Csc}[\theta]}{r^2\,u} \\ \frac{\beta 2\,\mathrm{Csc}[\theta]}{r\,u} & \frac{(-2\,v\,\beta 2+\gamma 2)\,\mathrm{Csc}[\theta]}{r\,u} & \frac{(2\,v\,\beta 1\,\beta 2-\beta 2\,\gamma 1-\beta 1\,\gamma 2)\,\mathrm{Csc}[\theta]}{r^2\,u} & \frac{(u+2\,\beta 2\,(v\,\beta 2-\gamma 2))\,\mathrm{Csc}[\theta]^2}{r^2\,u} \end{pmatrix}$$

NQS null frame

The NQS metric derives from the following null-frame $\{ l, n, m, \overline{m} \}$,

$$l == \partial_r - \vec{\beta}\,, \qquad n == \frac{\partial_z - v\left(\partial_r - \vec{\beta}\right) - \vec{\gamma}}{u}\,, \qquad m == \frac{\hat{\theta} - i\,\hat{\phi}}{\sqrt{2}}\,, \qquad \overline{m} == \frac{\hat{\theta} + i\,\hat{\phi}}{\sqrt{2}}$$

where $\langle l, n \rangle = -1$, $\langle m, \overline{m} \rangle = 1$, and all other frame-vector inner products are zero.

Vector fields on S^2 are associated with spin 1 scalar fields via,

$$\beta == \left\langle m, \vec{\beta} \right\rangle == \frac{\beta 1 - i\,\beta 2}{\sqrt{2}}\,, \qquad \vec{\beta} == \overline{m}\,\beta + m\,\overline{\beta}$$

In terms of the spin 1 fields β and γ,

$$\{g_{ij}\} == \begin{pmatrix} -2\,u\,v + 2\,\gamma\,\overline{\gamma} & -u + \overline{\beta}\,\gamma + \beta\,\overline{\gamma} & \frac{r\,(\gamma+\overline{\gamma})}{\sqrt{2}} & \frac{i\,r\,(\gamma-\overline{\gamma})\,\mathrm{Sin}[\theta]}{\sqrt{2}} \\ -u + \overline{\beta}\,\gamma + \beta\,\overline{\gamma} & 2\,\beta\,\overline{\beta} & \frac{r\,(\beta+\overline{\beta})}{\sqrt{2}} & \frac{i\,r\,(\beta-\overline{\beta})\,\mathrm{Sin}[\theta]}{\sqrt{2}} \\ \frac{r\,(\gamma+\overline{\gamma})}{\sqrt{2}} & \frac{r\,(\beta+\overline{\beta})}{\sqrt{2}} & r^2 & 0 \\ \frac{i\,r\,(\gamma-\overline{\gamma})\,\mathrm{Sin}[\theta]}{\sqrt{2}} & \frac{i\,r\,(\beta-\overline{\beta})\,\mathrm{Sin}[\theta]}{\sqrt{2}} & 0 & r^2\,\mathrm{Sin}[\theta]^2 \end{pmatrix}$$

The ð (eth) formalism

Tensor fields on S^2 are expressed as spin-weighted scalars, and covariant derivatives on S^2 are expressed in terms of the the spin raising/lowering operators ð and $\overline{ð}$,

$$ð\eta == \frac{\eta_\theta - i\,\mathrm{Csc}[\theta]\,\eta_\phi - \eta\,\mathrm{Cot}[\theta]\,\mathrm{spinWeight}[\eta]}{\sqrt{2}}$$

$$\nabla_\beta \eta == \beta\,\overline{ð}[\eta] + \overline{\beta}\,ð[\eta]$$

$$\mathrm{div}[\eta] == \overline{ð}[\eta] + ð[\overline{\eta}]$$

$$\mathrm{curl}[\eta] == i\left(\overline{ð}[\eta] - ð[\overline{\eta}]\right)$$

$$\mathrm{LapS2}[\eta] == \overline{ð}[ð[\eta]] + ð\left[\overline{ð}[\eta]\right]$$

$$\overline{ð}[ð[\eta]] - ð\left[\overline{ð}[\eta]\right] == \eta\,\mathrm{spinWeight}[\eta]$$

Numerical methods

- 8th order Dormand-Prince r-integrations.
- 8th order convolution splines for radial grid interpolation/differentiation.
- FFT for the θ and ϕ derivatives in ð, using the 2-torus as a double cover of S^2.
- 2/3-Orzag-like filtering to suppress non-linear aliasing.
- Projection from FFT coefficients to spin-weighted spherical harmonic coefficients to preserve uniform resolution over S^2 (& thereby eliminate polar instabilities).
- 4th order RK method for the z-integration.
- Solution of a 1st order elliptic equation on S^2 for the spin 1 field γ for each grid radius r and time-step z.

Further information

1997: R. Bartnik
Einstein equations in the null quasispherical gauge
Class. Quantum Grav., Vol 22, pp. 2185 — 2194.

2000: R. Bartnik and A. H. Norton
Numerical methods for the Einstein equations in null quasi-spherical coordinates
SIAM J. Sci. Comput., Vol 22, pp. 917— 950.

2002: R. A. Bartnik and A. H. Norton
Numerical experiments at null infinity
Chapter 16, pp. 313 — 326, in
The Conformal Structure of Space-Time: Geometry, Analysis, Numerics
J. Frauendiener and H. Friedrich (Eds.).

Our NQS work is reviewed in:

J. Winicour
Characteristic Evolution and Matching
Living Reviews in Relativity, 15, (2012), 2.

Robert’s Bondi metric

Introduction

Not previously published...

Robert's idea: Parametrize the 2-sphere part of a Bondi metric using a spin 2 field ξ, so that $\{m, \overline{m}\}$ are given by

$$m == a\,(m_0 - \overline{m}_0\,\xi)\,, \quad \text{where} \quad a == \frac{1}{\sqrt{1-\xi\,\overline{\xi}}}\,, \qquad m_0 == \frac{\hat{\theta} - i\,\hat{\phi}}{\sqrt{2}}\,.$$

The $\{m_0, \overline{m}_0\}$-frame can be interpreted as that of an undistorted reference S^2 that has standard polar coordinates $\{\theta, \phi\}$ with metric $r^2\,(d\theta \otimes d\theta + \sin[\theta]^2\, d\phi \otimes d\phi)$.

In deriving the following results I followed Robert's formulation of the NQS equations.
I did these calculations in Jan 2011 as an AEI postdoc while working for Robert.
Then I redid them for this talk.

For more on Bondi metrics, see T. Mädler and J. Winicour,
http://www.scholarpedia.org/article/Bondi-Sachs_Formalism

Coordinates and metric functions

Coordinates $\{z, r, \theta, \phi\}$.
The 3-surfaces $z = \text{const.}$ are ≈ forward null-cones, $\langle dz, dz \rangle = 0$.
The metric contains 6 arbitrary functions:

spin 0 fields U, V	(real)
spin 1 field γ	(complex)
spin 2 field ξ	(complex)

The r-coordinate lines are null-geodesics ⇒ $g_{rr} = \langle \partial_r, \partial_r \rangle = 0$,

$$\{g_{ij}\} == \begin{pmatrix} g_{zz} & g_{zr} & g_{z\theta} & g_{z\phi} \\ g_{rz} & g_{rr} & g_{r\theta} & g_{r\phi} \\ g_{\theta z} & g_{\theta r} & g_{\theta\theta} & g_{\theta\phi} \\ g_{\phi z} & g_{\phi r} & g_{\phi\theta} & g_{\phi\phi} \end{pmatrix}$$

$$== \begin{pmatrix} -U\,V + 2\,r^2\,\gamma\,\overline{\gamma} & -U & \frac{a\,r^2\,(\gamma + \overline{\gamma}\,(1+\xi) + \gamma\,\overline{\xi})}{\sqrt{2}} & -\frac{i\,a\,r^2\,(\overline{\gamma}\,(1-\xi) + \gamma\,(-1+\overline{\xi}))\,\text{Sin}[\theta]}{\sqrt{2}} \\ -U & 0 & 0 & 0 \\ \frac{a\,r^2\,(\gamma + \overline{\gamma}\,(1+\xi) + \gamma\,\overline{\xi})}{\sqrt{2}} & 0 & a^2\,r^2\,(1+\xi)\,(1+\overline{\xi}) & i\,a^2\,r^2\,(\xi - \overline{\xi})\,\text{Sin}[\theta] \\ -\frac{i\,a\,r^2\,(\overline{\gamma}\,(1-\xi) + \gamma\,(-1+\overline{\xi}))\,\text{Sin}[\theta]}{\sqrt{2}} & 0 & i\,a^2\,r^2\,(\xi - \overline{\xi})\,\text{Sin}[\theta] & -a^2\,r^2\,(1-\xi)\,(-1+\overline{\xi})\,\text{Sin}[\theta]^2 \end{pmatrix}$$

$$\sqrt{-\text{det}[g]} == r^2\,U\,\text{Sin}[\theta]$$

The inverse metric is,

$$\{g^{ij}\} == \begin{pmatrix} 0 & -\frac{1}{U} & 0 & 0 \\ -\frac{1}{U} & \frac{V}{U} & \frac{a\,(\gamma + \overline{\gamma}\,(1-\xi) - \gamma\,\overline{\xi})}{\sqrt{2}\,U} & \frac{i\,a\,(\gamma - \overline{\gamma}\,(1+\xi) + \gamma\,\overline{\xi})\,\text{Csc}[\theta]}{\sqrt{2}\,U} \\ 0 & \frac{a\,(\gamma + \overline{\gamma}\,(1-\xi) - \gamma\,\overline{\xi})}{\sqrt{2}\,U} & -\frac{a^2\,(1-\xi)\,(-1+\overline{\xi})}{r^2} & \frac{i\,a^2\,(-\xi + \overline{\xi})\,\text{Csc}[\theta]}{r^2} \\ 0 & \frac{i\,a\,(\gamma - \overline{\gamma}\,(1+\xi) + \gamma\,\overline{\xi})\,\text{Csc}[\theta]}{\sqrt{2}\,U} & \frac{i\,a^2\,(-\xi + \overline{\xi})\,\text{Csc}[\theta]}{r^2} & \frac{a^2\,(1+\xi)\,(1+\overline{\xi})\,\text{Csc}[\theta]^2}{r^2} \end{pmatrix}$$

The null frame

Robert's Bondi metric derives from the following null-frame $\{l, n, m, \overline{m}\}$,

$$l == \frac{\partial_r}{\sqrt{2}}\,, \qquad n == \frac{2\,\partial_z - \partial_r\,V - 2\,r\,(\overline{m}\,\gamma + m\,\overline{\gamma})}{\sqrt{2}\,U}\,, \qquad m == a\,(m_0 - \overline{m}_0\,\xi)\,,$$

$$\text{where} \quad a == \frac{1}{\sqrt{1-\xi\,\overline{\xi}}}\,, \qquad m_0 == \frac{\hat{\theta} - i\,\hat{\phi}}{\sqrt{2}}\,,$$

and where $\hat{\theta} = r^{-1}\partial_\theta$ and $\hat{\phi} = (r\,\text{Sin}[\theta])^{-1}\partial_\phi$.

The space-time metric (associated with the $\langle\,,\rangle$-inner product) is fixed by the conditions that $\langle l, n \rangle = -1$, $\langle m, \overline{m} \rangle = 1$, and that all other frame-vector inner products are zero.

Example: Schwarzschild

Metric functions,

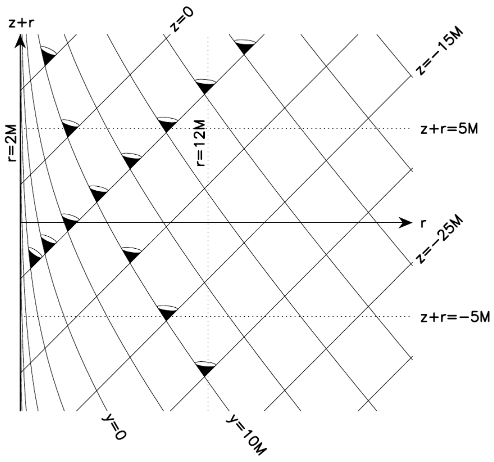
$$\left\{U \to 1,\; V \to 1 - \frac{2\,M}{r},\; \gamma \to 0,\; \xi \to 0\right\}$$

Null frame,

$$l == \frac{\partial_r}{\sqrt{2}}\,, \qquad n == \frac{2\,\partial_z - \partial_r\,\left(1 - \frac{2\,M}{r}\right)}{\sqrt{2}}\,, \qquad m == m_0\,,$$

Schwarzschild metric in Eddington–Finkelstein retarded coordinates (d'Inverno, p222),

$$\{g_{ij}\} = \begin{pmatrix} \frac{2M}{r}-1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2(\theta) \end{pmatrix}$$



The ð (eth) formalism with a background $ð_0$

The spin raising/lowering operators ð and $\bar{ð}$ correspond to covariant differentiation in the m and $\bar{m}$ directions, so they become more complicated on distorted spheres.

Let $ð_0$ denote eth on the reference S^2. The expression for $ð_0$ (same as in NQS) is,

$$ð_0\eta == \frac{\eta_\theta - i\,\mathrm{Csc}[\theta]\,\eta_\phi - \eta\,\mathrm{Cot}[\theta]\,\mathrm{spinWeight}[\eta]}{\sqrt{2}}$$

The ð in space-time can be calculated in terms of the background/reference $ð_0$ as

$$ð[\eta] == a\left(ð_0[\eta] - \xi\,\bar{ð}_0[\eta]\right) + \Gamma\,\eta\,\mathrm{spinWeight}[\eta]$$

where the spin 1 field Γ arises as a difference between connections,

$$\Gamma == \frac{1}{2}a^3\left(-\bar{\xi}\,ð_0[\xi] - \xi\,ð_0[\bar{\xi}] + \bar{ð}_0[\xi] + \frac{\bar{ð}_0[\xi]}{a^2} + \xi^2\,\bar{ð}_0[\bar{\xi}]\right)$$

The above formulas can be used to numerically evaluate $ð\eta$.
E.g. FFT calculation of η_θ, η_ϕ for $ð_0\eta$, and FFT calculation of ξ_θ, ξ_ϕ for Γ.

The background operator $ð_0$ will not appear in any further eqns.

The spin 1 field Γ does appear later. In particular, Γ appears in $\hat{G}_{ln}$, which involves the Gaussian curvature of the coordinate 2-spheres.

Notation

As before, we write

$$\nabla_\beta\eta == \beta\,\bar{ð}[\eta] + \bar{\beta}\,ð[\eta]$$

$$\mathrm{div}[\eta] == \bar{ð}[\eta] + ð[\bar{\eta}]$$

$$\mathrm{curl}[\eta] == i\left(\bar{ð}[\eta] - ð[\bar{\eta}]\right)$$

$$\mathrm{LapS2}[\eta] == \bar{ð}[ð[\eta]] + ð\left[\bar{ð}[\eta]\right]$$

$$\bar{ð}[ð[\eta]] - ð\left[\bar{ð}[\eta]\right] == \eta\,\mathrm{spinWeight}[\eta]$$

The dot and cross notation is also useful,

$$S\times\xi == i\left(\bar{S}\,\xi - S\,\bar{\xi}\right)$$

$$\alpha\cdot\beta == \alpha\,\bar{\beta} + \beta\,\bar{\alpha}$$

Connection variables: S, Q, $Q^{\pm}$, J, K

We introduce connection variables { S, Q, $Q^{\pm}$, J, K }. These are more-or-less Ricci rotation coefficients for the null-frame. They are spin-weighted scalar fields (J is real, the others complex),

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spinWeight[{S, Q, Q⁺, Q⁻, J, K}]
{2, 1, 1, 1, 0, 2}
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They are defined in terms of 1st derivs of the metric variables { U, V, γ, ξ },

$$S == -a^2\,r\,\xi_r$$

$$Q == \frac{i\,r^3\,\gamma\,S\times\xi - 2\,r^3\,(S\,\bar{\gamma} + r\,\gamma_r)}{2\,U}$$

$$Q^{\pm} == Q \pm \frac{r^2\, \mathrm{eth}[U]}{U}$$

$$J == \frac{V}{U} + \frac{r\, \mathrm{div}[\gamma]}{U}$$

$$K == \frac{S\,V}{2} - r\, \mathrm{eth}[\gamma] + a^2\, r\, \xi_z$$

Relation to NP coefficients

For those familiar with NP/GHP notation:

The connection variables are equivalent to $\{\sigma, \overline{\tau'}, \tau, \overline{\rho'}, \overline{\sigma'}\}$,

$$\left\{\sigma \to \frac{S}{\sqrt{2}\, r},\ \overline{\tau'} \to -\frac{Q^{+}}{2\, r^3},\ \tau \to \frac{Q^{-}}{2\, r^3},\ \overline{\rho'} \to \frac{J}{\sqrt{2}\, r},\ \overline{\sigma'} \to -\frac{\sqrt{2}\, K}{r\, U}\right\}$$

$$\left\{S \to \sqrt{2}\, r\, \sigma,\ Q^{+} \to -2\, r^3\, \overline{\tau'},\ Q^{-} \to 2\, r^3\, \tau,\ J \to \sqrt{2}\, r\, \overline{\rho'},\ K \to -\frac{r\, U\, \overline{\sigma'}}{\sqrt{2}}\right\}$$

and $\kappa = 0$, so the outward null vector field l satisfies the geodesic equation.

E. Newmann and R. Penrose,
http://www.scholarpedia.org/article/Spin-coefficient_formalism

The solution algorithm

The *characteristic initial value problem* is to solve the Einstein equations outside of a worldtube $(r = r_0)$ with initial data on a forward null-surface $(z = z_0)$ from the worldtube. To solve this problem:

Step 1 — calculate the shear S

Suppose that ξ is given as initial data on the null-surface $z = z_0$.

By numerical differentiation of ξ in the r-direction, calculate the shear,

$$S == -a^2\, r\, \xi_r$$

where,

$$a == \frac{1}{\sqrt{1 - \xi\, \overline{\xi}}}$$

Step 2 — integrate the hypersurface equations

Assuming that we somehow know what data should be used at the tube, $r = r_0$:

Integrate the following *hypersurface equations* from $r = r_0$ to $r = \infty$.
For a vacuum soln, we set all Einstein frame components $\hat{G}_{ab} = 0$,

$$U_r == \frac{S\, \overline{S}\, U}{r} + \frac{r\, U\, \hat{G}_{ll}}{\chi^2}$$

$$Q^{-}{}_r == \frac{\overline{Q^{-}}\, S}{r} + \frac{i\, Q^{-}\, S \times \xi}{2\, r} - \frac{4\, r\, \mathrm{eth}[U]}{U} + 2\, r\, \mathrm{ethb}[S] + \frac{2\sqrt{2}\, r^3\, \hat{G}_{lm}}{\chi}$$

$$J_r == -\frac{Q^{+}\, \overline{Q^{+}}}{2\, r^5} - \frac{\mathrm{div}[Q^{+}]}{2\, r^3} + \frac{1 - J\left(1 + S\, \overline{S}\right) + \mathrm{div}[\Gamma]}{r} + r\left(-\frac{J\, \hat{G}_{ll}}{\chi^2} - \hat{G}_{ln}\right)$$

$$K_r == -\frac{J\, S\, U}{2\, r} + \frac{i\, K\, S \times \xi}{r} + U\left(\frac{Q^{+2}}{4\, r^5} + \frac{\mathrm{eth}[Q^{+}]}{2\, r^3} + \frac{1}{2}\, r\, \hat{G}_{mm}\right)$$

...these can be integrated simultaneously as a system, or sequentially as ordered.

Step 3 — integrate the eqn for γ

This eqn follows from the defining eqn for Q.
Integrate from $r = r_0$ to $r = \infty$,

$$r\, \gamma_r == -\frac{Q\, U}{r^3} - S\, \overline{\gamma} + \frac{1}{2}\, i\, \gamma\, S \times \xi$$

Step 4 — calculate V

This eqn follows from the defining eqn for J,

$$V == J\, U - r\, \mathrm{div}[\gamma]$$

Step 5 — calculate ξ_z

This eqn follows from the defining eqn for K,

$$a^2\,\xi_z == \frac{K}{r} - \frac{S\,V}{2\,r} + \text{eth}[\gamma]$$

Step 6 — evolve ξ

Using the computed value for ξ_z, time-step ξ to the next null-surface at $z = z_0 + \delta z$.

Then repeat from Step 1.

(Using RK4 would actually require calculating ξ_z four times per time-step.)

The boundary equations

Some of the initial data for the r-integrations corresponds to outgoing gravitational radiation emitted from the system that the worldtube encloses (e.g. a 3+1 simulation, or a black hole).

The rest of the initial data for the r-integrations is constructed using *boundary equations* for J_z and $Q^+{}_z$, which follow from $\hat{G}_{nn} = 0$, and $\hat{G}_{nm} = 0$,

$$J_z == \frac{2\,K\,\bar{K}}{r\,U} + \frac{3\,J^2\,U}{2\,r} - \frac{J\,\nabla_Q U}{2\,r^3} - \frac{\nabla_{Q^\cdot} V}{2\,r^3} + \nabla_\gamma J - \frac{V\,\text{div}[Q]}{2\,r^3} + J\,V\left(-\frac{3}{2\,r} - \frac{S\,\bar{S}}{2\,r} - \frac{r\,\hat{G}_{ll}}{2\,\chi^2}\right) + \text{div}[\gamma]\left(-\frac{1}{2} + \frac{Q^+\,\overline{Q^+}}{4\,r^4} - \frac{\text{div}[Q^-]}{4\,r^2} - \frac{\text{div}[\Gamma]}{2} + \frac{1}{2}\,r^2\,\hat{G}_{ln}\right) + \frac{1}{2}\,r\,U\,\chi^2\,\hat{G}_{nn} - \frac{\text{LapS2}[V]}{2\,r}$$

$Q^+{}_z ==$ Something involving $\hat{G}_{nm}$ (not yet calculated)