Memory now and then:

Current developments in mathematical general relativity

ASGRG Meeting, ANU, September 2024

Volker Schlue

University of Melbourne

THE UNIVERSITY OF MELBOURNE

POSTERA CRESCAM LAUDE

VOLUME 67, NUMBER 12 PHYSICAL REVIEW LETTERS 16 SEPTEMBER 1991

Nonlinear Nature of Gravitation and Gravitational-Wave Experiments

Demetrios Christodoulou
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(Received 17 December 1990; revised manuscript received 20 June 1991)

It is shown that gravitational waves from astronomical sources have a nonlinear effect on laser interferometer detectors on Earth, an effect which has hitherto been neglected, but which is of the same order of magnitude as the linear effects. The signature of the nonlinear effect is a permanent displacement of test masses after the passage of a wave train.

PACS numbers: 04.30.+x, 04.80.+z

https://math.ethz.ch/news-and-events/news/d-math-news/2018/05/farewell-to-demetrios-christodoulou.html

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D-MATH NEWS

A farewell to Demetrios Christodoulou

This coming Wednesday, 16 May 2018, Professor Demetrios Christodoulou will give his farewell lecture at ETH Zurich – looking back on a journey through mathematics and physics that has spanned nearly half a century.

14.05.2018 by Andreas Trabesinger Read

"Christodoulou's work combines a deep understanding of [...] physics with brilliant mathematical technique. This has allowed him to resolve central problems that have resisted progress for generations." So wrote the Shaw Prize Selection Committee as it awarded Demetrios Christodoulou the 2011 Prize in Mathematical Sciences, shared with the American Mathematician Richard S. Hamilton [1]. The Shaw Prize – often referred to as the "Nobel of the East" – was one of many awards and distinctions bestowed on Christodoulou [2], for work that not only explored the interface between mathematics and physics, but ventured deeply into both disciplines.

Christodoulou's journey so far through the two disciplines has been an extraordinary one. Born in 1951 in Athens, to parents neither of whom had a formal higher education, he

Demetrios Christodoulou (far right) in discussion with Kip Thorne (2017 Phyiscs Nobel laureate) at the École de Physique des Houches 1972; on the left are Yuval Ne'eman and Bryce DeWitt. (Photo: Wikimedia / A. T. Service, CC BY-SA 3.0)

Demetrios Christodoulou

Farewell lecture

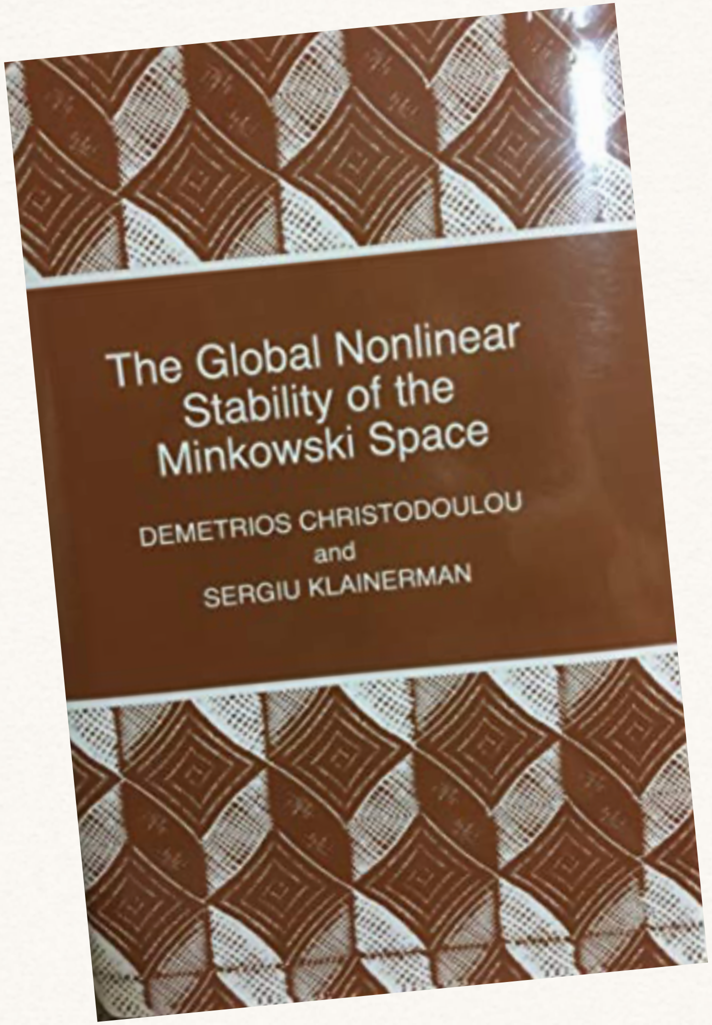
A personal experience through mathematics and physics

Prof. Demetrios Christodoulou
16 May 2018
Video

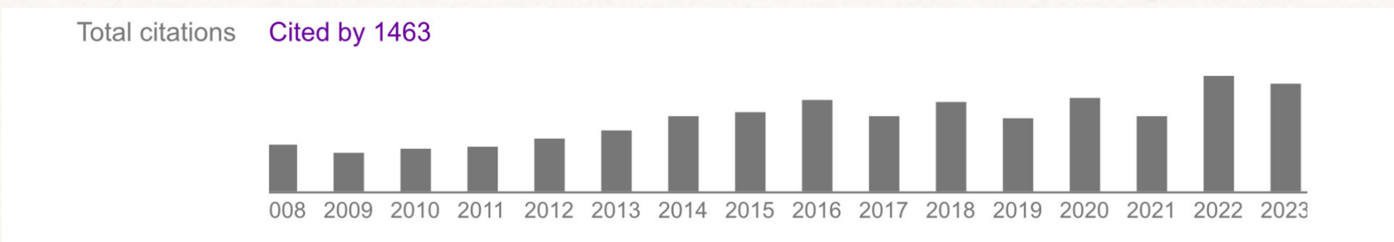
- *The asymptotics of the gravitational field and the memory effect*, **Dipl.phys. Thesis**, ETH Zurich (2008), [expository, after D. Christodoulou, *Nonlinear Nature of Gravitation and Gravitational-Wave Experiments*, Phys. Rev. Letters (67), no 12, pp. 1486–1489, (1991)], 48 pages.

  PDF

https://blogs.unimelb.edu.au/volker-schlue/#tab13



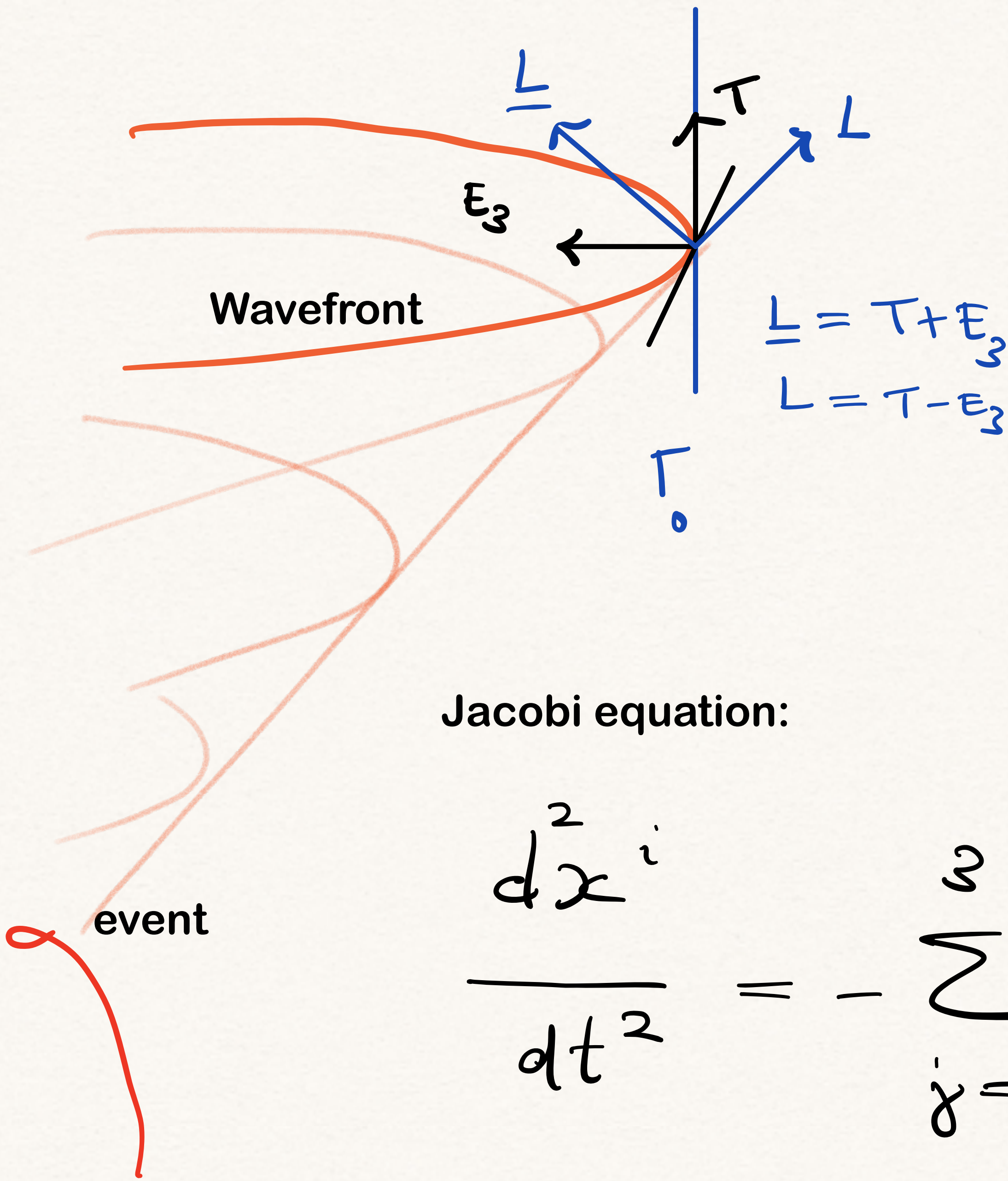
Citations in the decades 1993-2003: 218 2003-2013: 408 2013-2023: 831

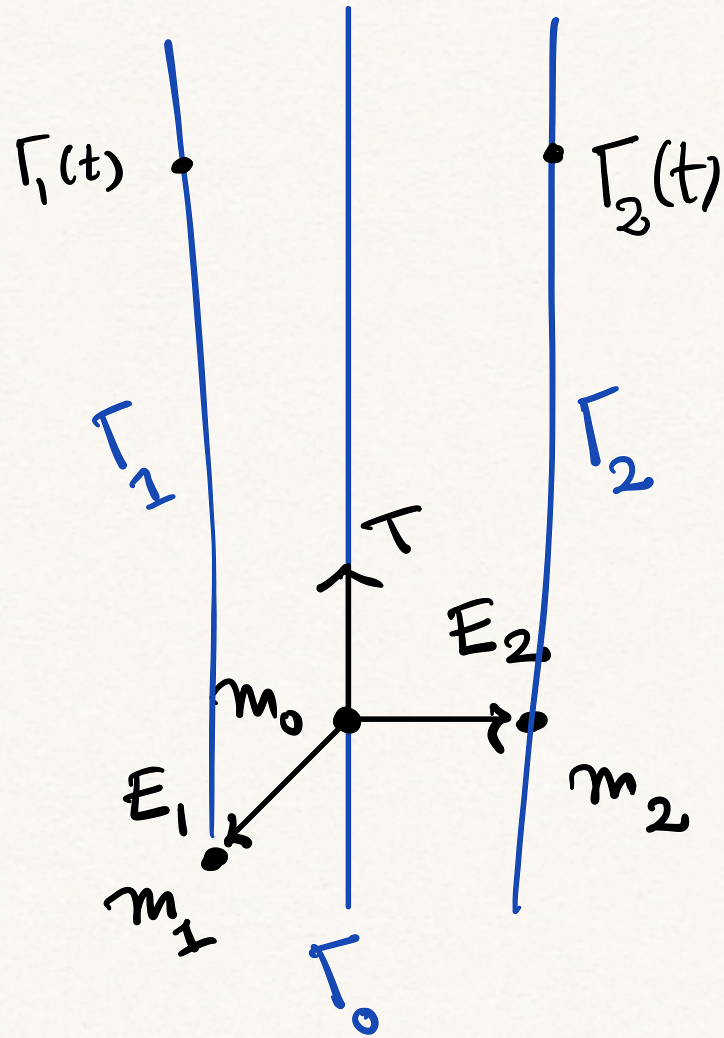


Maths Genealogy Project: Demetrios Christodoulou has 13 students, and 48 descendants;

Sergiu Klainerman has 23 students, and 77 descendants.

(Idealised) Gravitational wave experiment

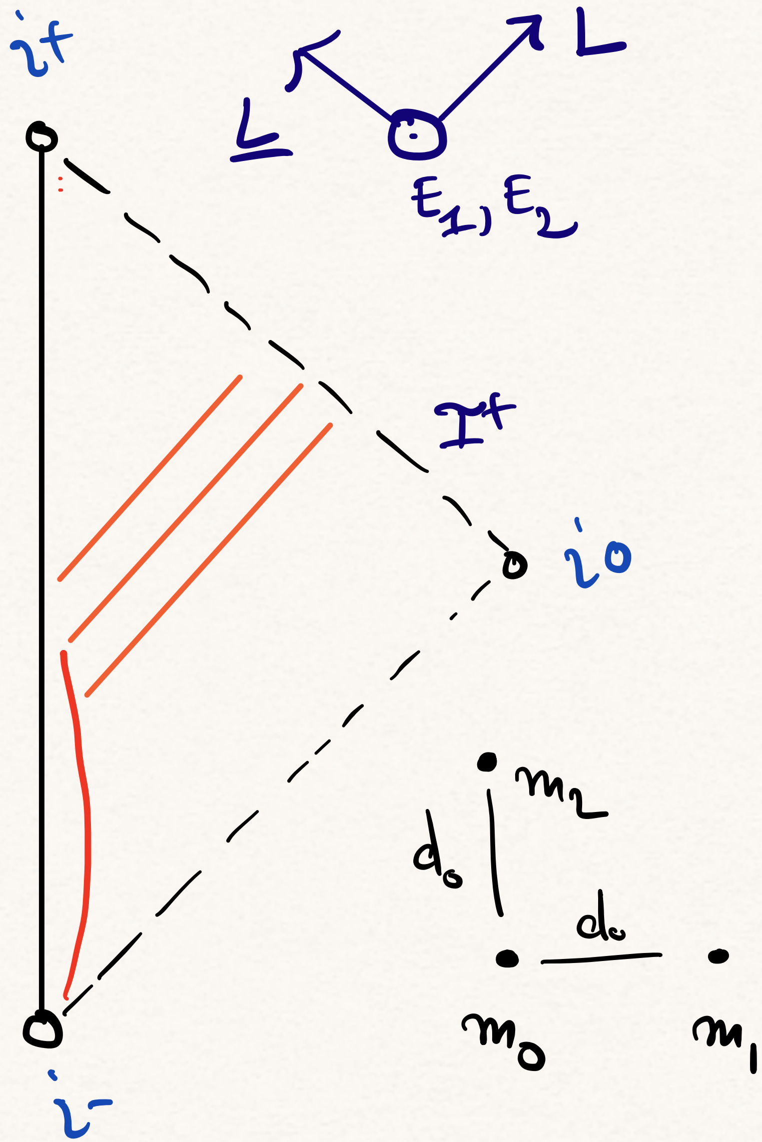




Jacobi equation:

$$\frac{d^2 x^i}{dt^2} = - \sum_{j=0}^{3} R(E_i, T, E_j, T)\Big|_{\Gamma_0(t)} x^j$$

Asymptotics of the gravitational field



$$\underline{\mathcal{L}}_{AB} = \mathcal{R}(E_A, \underline{L}, E_B, \underline{L})$$

$$\sim \frac{A_{AB}}{r}$$

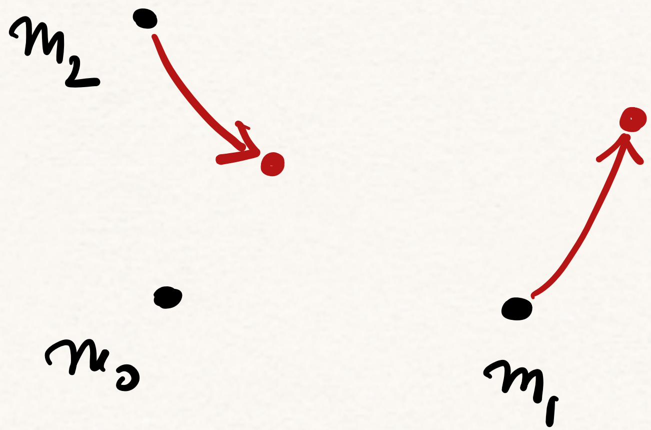
$$\frac{d^2 x^A_{(B)}}{dt^2} \simeq -\frac{d_0}{4r} A_{AB}{}^{(t)}$$

$x^A_{(B)}$: Coords $A = 1, 2,$ of mass $B = 1, 2.$

r : distance from source, d : distance between test masses in detector

Memory effect

$$\Delta x^A_{(B)}(\infty) \neq 0.$$



$$\frac{d^2 x^A_{(B)}}{dt^2} = -\frac{d_0}{4r} A_{AB}(t)$$

$$\frac{dx^A_{(B)}}{dt} = \frac{d_0}{2r} \Xi_{AB}, \quad \Xi_{AB}(t) = -\frac{1}{2}\int_{-\infty}^{t} A_{AB}(u)\,du$$

$\Xi_{AB}(\infty)$: final kick velocity

$$\Delta x^A_{(B)}(t) = x^A_{(B)}(t) - x^A_{(B)}(-\infty) = \frac{d_0}{2r}\int_{-\infty}^{t} \Xi_{AB}(u)\,du$$

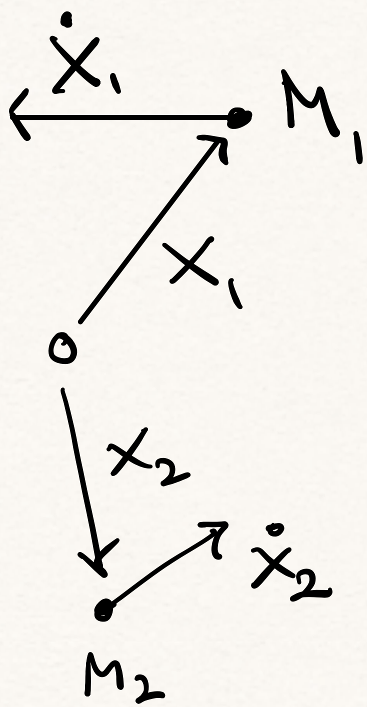
Linear theory

Quadrupole formula:

$$\mathrm{A} \sim \frac{d^4}{dt^4} Q$$

where
$$Q^{ij} = \sum_\alpha M_\alpha X^i_\alpha X^j_\alpha$$

Einstein '18, Ehlers et al '76; derived in slow motion and weak field approximation, see eg. Poisson & Will '14 for discussion. Confirmed experimentally since Taylor '78.



$$\Xi \sim \frac{d^3}{dt^3} Q$$

$$\Delta x^A_{(B)} \sim \frac{d^2}{dt^2} Q(\infty) - \frac{d^2}{dt^2} Q(-\infty)$$

$$\frac{d^2}{dt^2} Q \sim \sum_\alpha M_\alpha \dot{X}^i_\alpha \dot{X}^j_\alpha \sim \text{kinetic energy.}$$

Zel'dovich and Polnarev '74, Braginsky and Thorne '87.

Non-linear theory

$$\Delta x^{A}_{(B)} = \frac{d_0}{r}\left(\Sigma^{+} - \Sigma^{-}\right)_{AB}$$

$$\partial_u \Sigma = -\frac{1}{2}\,\Xi\,, \qquad \partial_u \Xi = -\frac{1}{2}\,A\,,$$

Total energy radiated in a given direction:

$$F(\xi) = \frac{1}{8}\int_{-\infty}^{\infty} |\Xi|^2(u,\xi)\,du\,, \quad \xi \in S^2$$

Christodoulou '91:

$$\left(\Sigma^{+}-\Sigma^{-}\right)(\xi) = \frac{-1}{2\pi}\int_{|\xi'|=1}(F - F_{[1]})(\xi')\,\frac{\langle X,\xi'\rangle\langle Y,\xi'\rangle - \frac{1}{2}\langle X,Y\rangle|\Pi\xi'|^2}{1-\langle\xi,\xi'\rangle}\,d\mu_{\gamma^0}(\xi')\,.$$

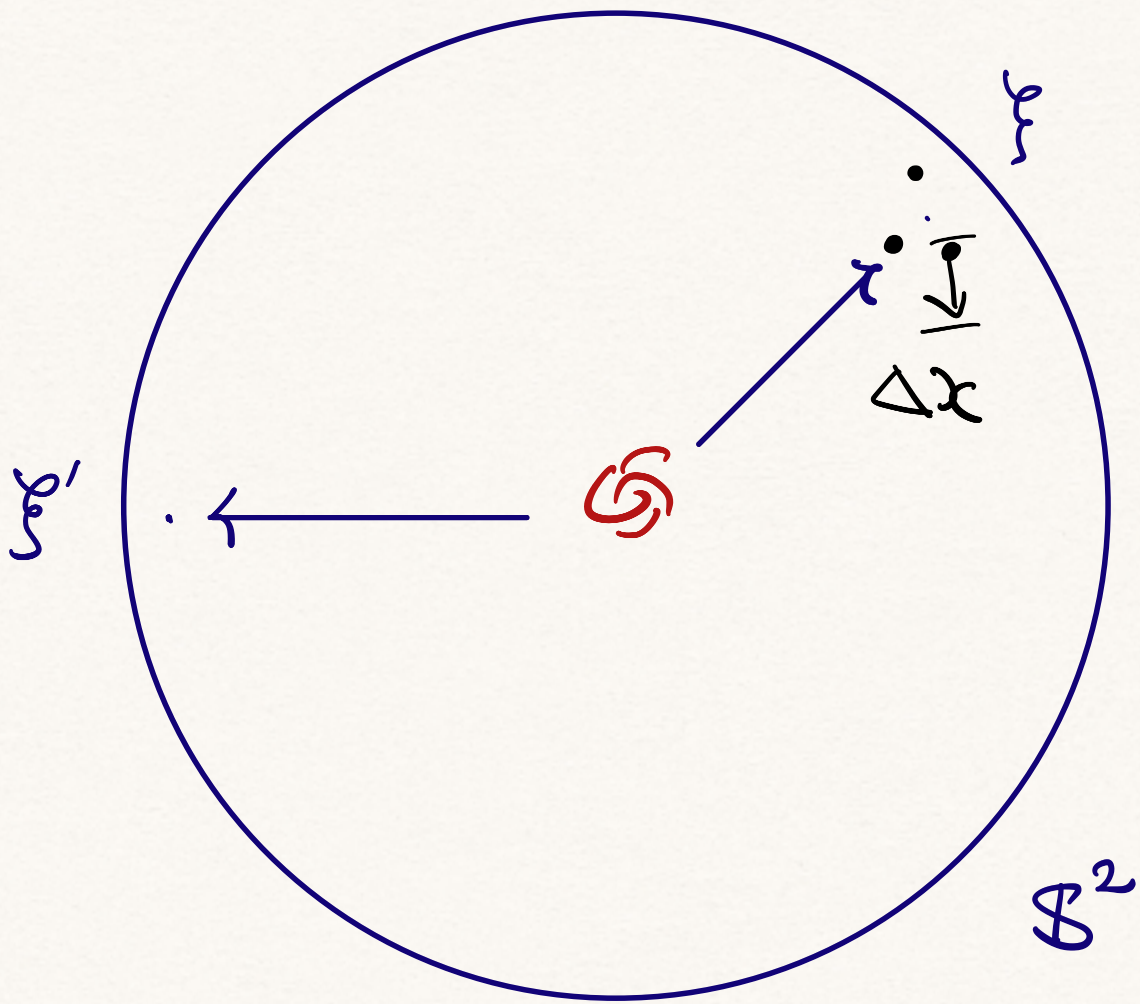
When matter (i.e., electromagnetic or neutrino) radiation is present then if T is the energy tensor of matter

"nonlinea

angles fr

Remarks on the Christodoulou memory effect

The effect is non-linear and non-local:



$$\Delta x \sim \Sigma^+ - \Sigma^-$$

$$(\Sigma^+ - \Sigma^-)(\xi) \sim$$

$$\sim \int_{S^2} F(\xi')\, K(\xi,\xi')\, dS(\xi')$$

$$F(\xi) = \int |\Xi|^2(\xi)\, du$$

Developments in mathematical GR since the 90's

New proofs of the stability of Minkowski space:

Lindblad-Rodnianski '05, '08; Bieri '09; Lindblad '17; Hintz-Vasy '20; Graf'20, Shen '22.

With matter models:

Speck '14; LeFloch-Ma '15, Wang '16; Taylor '16, Lindblad-Taylor '17; Fajman-Joudioux-Smulevici '17; Lindblad - Kauffman '23.

Much progress on fluid models and free boundary problems; and black hole stability problems: discussion for another time.

Cosmological setting:

Ringström '08; Rodnianski-Speck '13; Oliynyk '16, Fajman-Oliynyk-Wyatt '21; Hadzic-Jang '18, '20; Hintz-Vasy '18; Fournodavlos-Schlue '24.

Scattering: Lindblad-Soffer '05; Wang '10; Lindblad-Schlue '18, '24; He '21; Yu '22, '24.

Peeling: Kehrberger '21, '22, '24; Kehrberger-Masaood '24.

Memory effect: Bieri-Chen-Yau '12, Bieri-Garfinkle '13, '14, '23; Bieri '22.

Ehlers, Goldberg, Havas, Rosenbaum '76. "Comments on gravitational radiation damping and energy loss in binary systems"

Little is known rigorously about gravitational radiation from localized matter distributions. On the assumption of asymptotically flat spacetime in the vicinity of null infinity it has been shown (Sachs 1962; Newman and Unti 1962) that (except in certain special cases) null rays coming from an isolated matter distribution shear as they approach null infinity as seen in an asymptotically Minkowskian frame. The square of the rate of change of shear determines the rate of radiation of gravitational energy. What is missing is a connection between the shear and the observed motion of stellar systems, in particular their quadrupole moments, and thus it is not possible to translate the rigorously known result involving the shear into a numerically testable statement.

In order to obtain such a testable statement, it has been necessary to resort to approximation methods which are based on an iteration or expansion of the deviation of the physical metric tensor from the Minkowski flat space metric. These attempts have not yet arrived at a consistent solution. Here we shall outline

The *problem* is to determine approximately spacetime models (M, g, T) which fulfill the following conditions:

I) They satisfy *Einstein's equation*

$$G = 8\pi T\,, \tag{1}$$

and thus also its consequence

$$\nabla\cdot T = 0\,. \tag{2}$$

II) They correspond to a physically reasonable *model of the sources* (bodies). This can be achieved either by specifying the functional dependence of the stress-energy-momentum tensor T on some matter variables m_A and the metric g,

$$T = T(m_A, g) \tag{3}$$

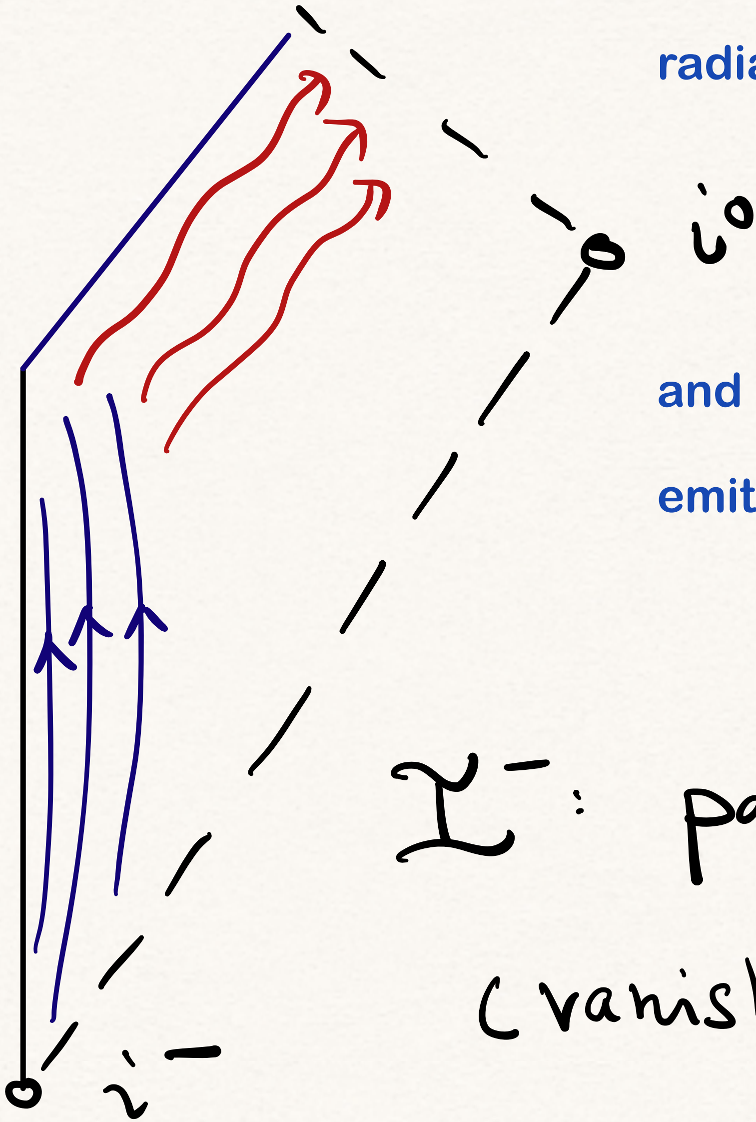
III) They satisfy a *condition for absence of incoming radiation*. The precise form of the boundary condition (III) is not known and can perhaps be formulated only once a satisfactory solution of the whole problem has been obtained. Some such condition, however, is certainly needed to make the problem mathematically well defined and to express the physical assumption that the system has indeed been isolated except during some unknown prehistory. It is not known whether nonstationary spacetimes exist which possess a past null infinity $\mathscr{I}^-$ (in the sense of Penrose 1964), and which of them, if any, are free of incident radiation; in model theories Leipold (1976) has shown that it depends on the early motion of the sources (for $t \to -\infty$) whether retarded fields do or do not contain incident radiation. In view of these facts and the presently accepted view that the existence of $\mathscr{I}$ is an essential part of the definition of an isolated system, it appears to be premature to claim that gravitational radiation from isolated material systems is theoretically well understood, even in principle.

Scattering problem:

Construct solutions to Einstein's equations

from past null infinity, satisfying the no-incoming

radiation condition,



and study the outgoing radiation

emitted from matter coming in from time-like infinity.

$\mathcal{I}^-$: past null infinity

(vanishing shear: $r\hat{\chi} \longrightarrow 0$)

Vlasov matter

Statistical ensemble of self-gravitating particles which interact only indirectly through the Einstein equations.

small energy density

$\Rightarrow$ global existence.

Ann. PDE (2017) 3:9
DOI 10.1007/s40818-017-0026-8

CrossMark

The Global Nonlinear Stability of Minkowski Space for the Massless Einstein–Vlasov System

Martin Taylor[1]

Arch. Rational Mech. Anal. 235 (2020) 517–633
Digital Object Identifier (DOI) https://doi.org/10.1007/s00205-019-01425-1

Check for updates

Global Stability of Minkowski Space for the Einstein–Vlasov System in the Harmonic Gauge

HANS LINDBLAD & MARTIN TAYLOR

ANALYSIS AND PDE
Vol. 14, No. 2, 2021
https://doi.org/10.2140/apde.2021.14.425

msp

THE STABILITY OF THE MINKOWSKI SPACE FOR THE EINSTEIN–VLASOV SYSTEM

DAVID FAJMAN, JÉRÉMIE JOUDIOUX AND JACQUES SMULEVICI

Also Vlasov-Poisson, and Vlasov-Maxwell: Smulevici ‘16, Smulevici-Fajman ‘17, Flynn-Ouyang-Pausader-Widmayer ‘23, Bigorne ‘22, ‘23, Bigorne-Velozo Ruiz ‘24

Question: Is the quadrupole formula valid in the context of the Einstein-Vlasov system?

1. Scattering solutions to the Einstein vacuum equations in harmonic coordinates?
2. Scattering data for matter at time-like infinity?
3. Proof of the quadrupole formula without representation formula?

yes, partially, there is progress for

1. – Lindblad-Schlue '17, '23

2. – Schlue-Taylor '24

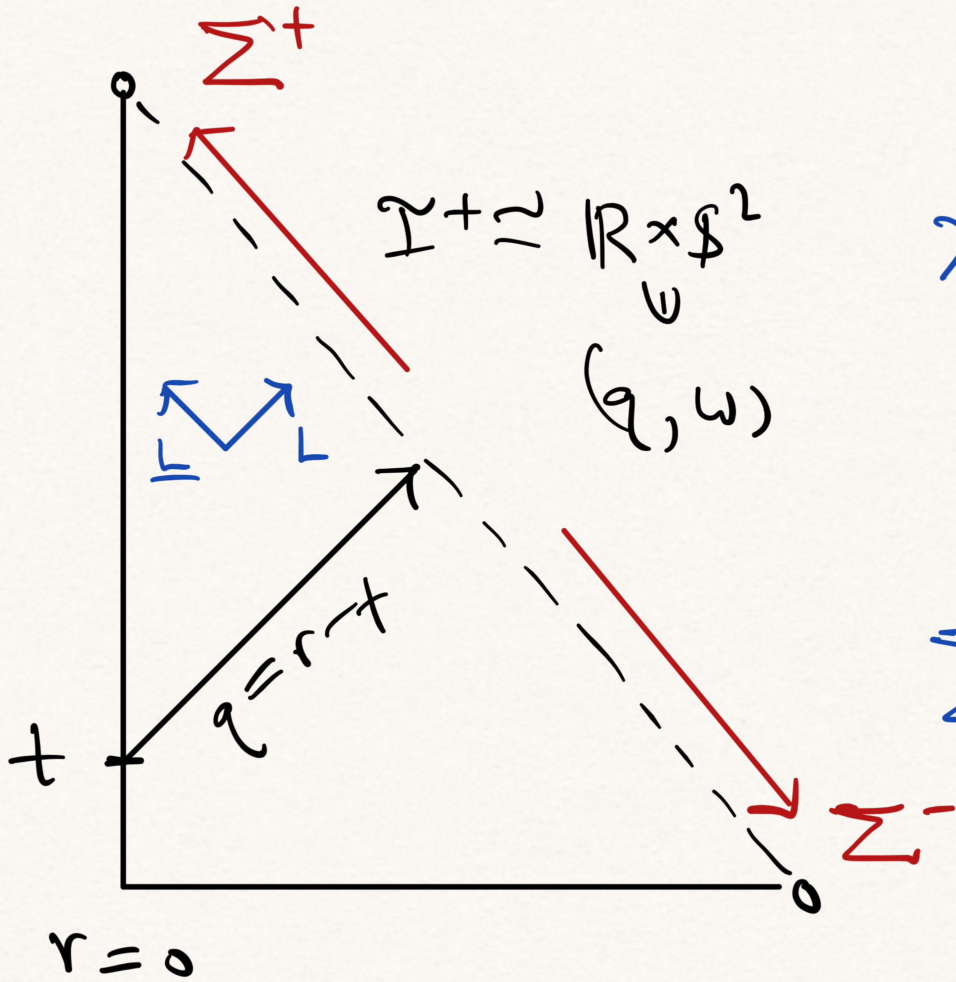
Maybe easier: Dipole formula for Vlasov-Maxwell?

Cf. Bauer, Kunze, Rein, Rendall '06

Relation of memory effect to scattering problems:

Geometric interpretation of $\Sigma^+ - \Sigma^-$.

<u>Shear:</u> $$\Sigma(q) = \lim_{\substack{q = r-t, \\ r\to\infty}} r^2 \hat{\chi}$$



$$\chi \sim L g, \quad g(L, L) = 0$$

$\hat{\chi}$ trace-free part.

$$\Sigma^{\pm} = \lim_{q \to \mp\infty} \Sigma(q)$$

Relation of memory effect to scattering problems:

Einstein vacuum equations in wave coordinates:

$$\Box^* h_{\mu\nu} = F_{\mu\nu}(\partial h, \partial h) \,, \qquad g = m + h$$

This system has weak null structure:

Example:

$$\Box \varphi = (\partial_t \psi)^2$$

$$\Box \psi = -(\partial_t \psi)^2 + |\nabla \psi|^2$$

Radiation field:

$$\psi(t, r\omega) \sim \frac{\mathcal{F}_0(r-t, \omega)}{r} \qquad (r \sim t)$$

Relation of memory effect to scattering problems:

Analogues for scalar equations: cubic sources

$$\Box \phi = \frac{m(r-t,\omega)}{r^3}$$

$$\phi \sim \frac{\mathcal{F}_0(r-t,\omega)}{r} + \frac{\mathcal{F}_1(r-t,\omega)}{r^2}$$

Analogue of shear:

$$\mathcal{F}_1(q,\omega) = -\lim_{\substack{q=r-t,\\ r\to\infty}} r^2(\partial_t+\partial_r)(r\phi)$$

$$\mathcal{F}_1^+(\omega) - \mathcal{F}_1^-(\omega) = \int_{\mathbb{S}^2} M(\sigma)\, dS(\sigma), \quad M(\sigma) = \int_{-\infty}^{\infty} m(q,\sigma)\, dq$$

Lindblad-Schlue '23

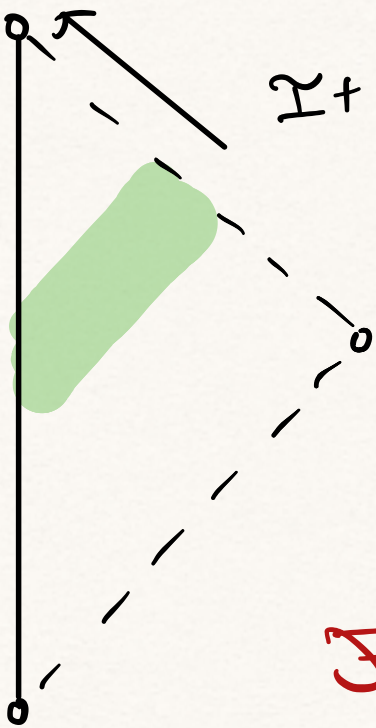
Relation of memory effect to scattering problems:

Analogues for scalar equations: quadratic sources

$$\Box \varphi = \frac{n(r-t,\omega)}{r^2}$$

$$\varphi(t, r\omega) \sim \frac{1}{r} F_{01}(r-t,\omega) \ln \frac{\langle r+t \rangle}{\langle r-t \rangle}$$

$$+ \frac{1}{r} F_0(r-t,\omega)$$



$$F_{01}(q) \longrightarrow N(\omega) = \int_{-\infty}^{\infty} n(q,\omega)\,dq$$

and

$$F_0^+ - F_0^- = \int_{S^2} \frac{N(\sigma) - N(\omega)}{1 - \langle \sigma, \omega \rangle}\, dS(\sigma)$$

Open problem beyond validity of the quadrupole formula:

Prove the quadrupole formula for the Einstein-Vlasov system for an ensemble of incoming particles with initially non-relativistic relative velocities under the no-incoming radiation condition, and prove that:

$$\Xi \sim \frac{d^3}{dt^3} Q^{TT} \sim \frac{1}{t^2} \qquad (t \to -\infty)$$

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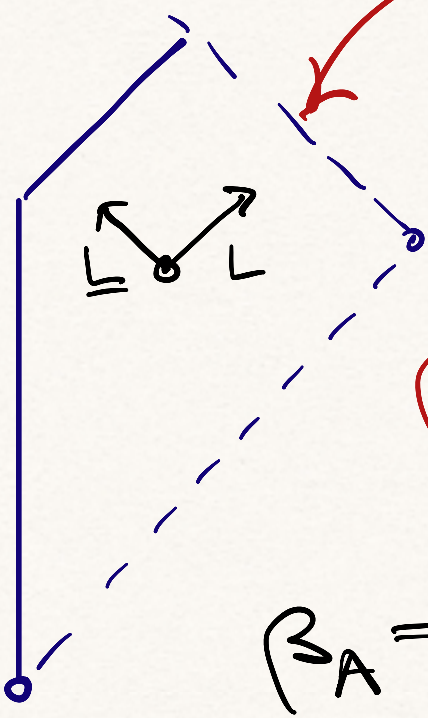
THE GLOBAL INITIAL VALUE PROBLEM IN GENERAL RELATIVITY

DEMETRIOS CHRISTODOULOU
Department of Mathematics, Princeton University, Princeton University, Princeton, New Jersey 08544, USA

The lecture, in part, discusses the progress that has so far been accomplished in the investigation of the global initial value problem in general relativity and, in addition, addresses the fundamental open problems that remain to be solved. In particular the development of the theory of gravitational radiation is discussed. Also, the problem of the formation and structure of spacetime singularities is discussed in relation to the fundamental issue of predictability. Here, the lecture touches on what can be learned from the study of analogous problems in fluid mechanics.

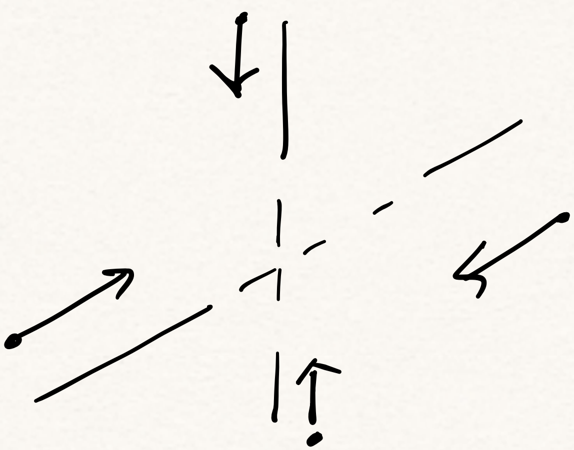
smooth ?

peeling ?



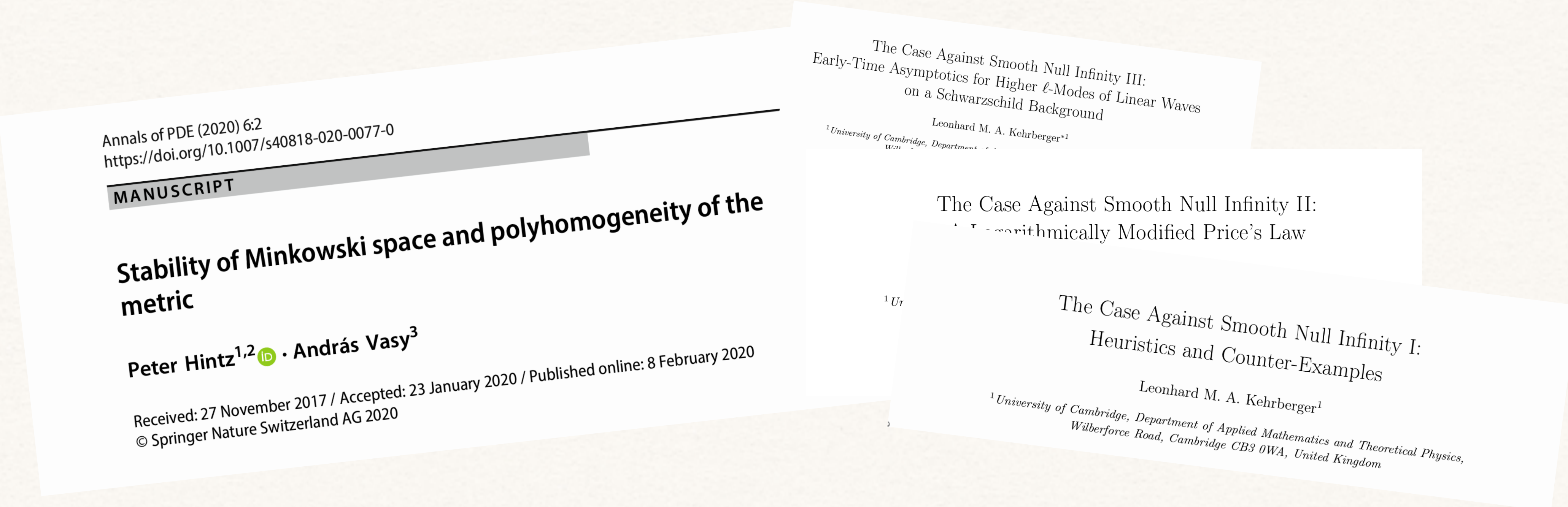
$$\beta_A \sim \frac{B_A}{r^4} \ln\left(\frac{\langle t+r\rangle}{\langle t-r\rangle}\right)$$

$$\beta_A = \frac{1}{2} R(E_A, L, \underline{L}, L)$$



Recent activity:

Polyhomogeneity: Hintz-Vasy ‘17, Lindblad ‘17, Lindblad-He ‘19.


Annals of PDE (2020) 6:2
https://doi.org/10.1007/s40818-020-0077-0

MANUSCRIPT

Stability of Minkowski space and polyhomogeneity of the metric

Peter Hintz[1,2] · András Vasy[3]

Received: 27 November 2017 / Accepted: 23 January 2020 / Published online: 8 February 2020
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The Case Against Smooth Null Infinity III:
Early-Time Asymptotics for Higher ℓ-Modes of Linear Waves on a Schwarzschild Background

Leonhard M. A. Kehrberger*[1]

The Case Against Smooth Null Infinity II:

The Case Against Smooth Null Infinity I:
Heuristics and Counter-Examples

Leonhard M. A. Kehrberger[1]

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„Modified“ Scattering: Lindblad-Schlue ‘23; He ‘21; Yu ‘22, ‘24, Schlue-Taylor ‘24

Failure of smoothness: Kehrberger ‘21, ‘22, ‘24; Kehrberger-Masaood ‘24.

Memory effect: Bieri-Chen-Yau ‘12, Bieri-Garfinkle ‘13, ‘14, ‘23; Bieri ‘22.

— The end, thank you!