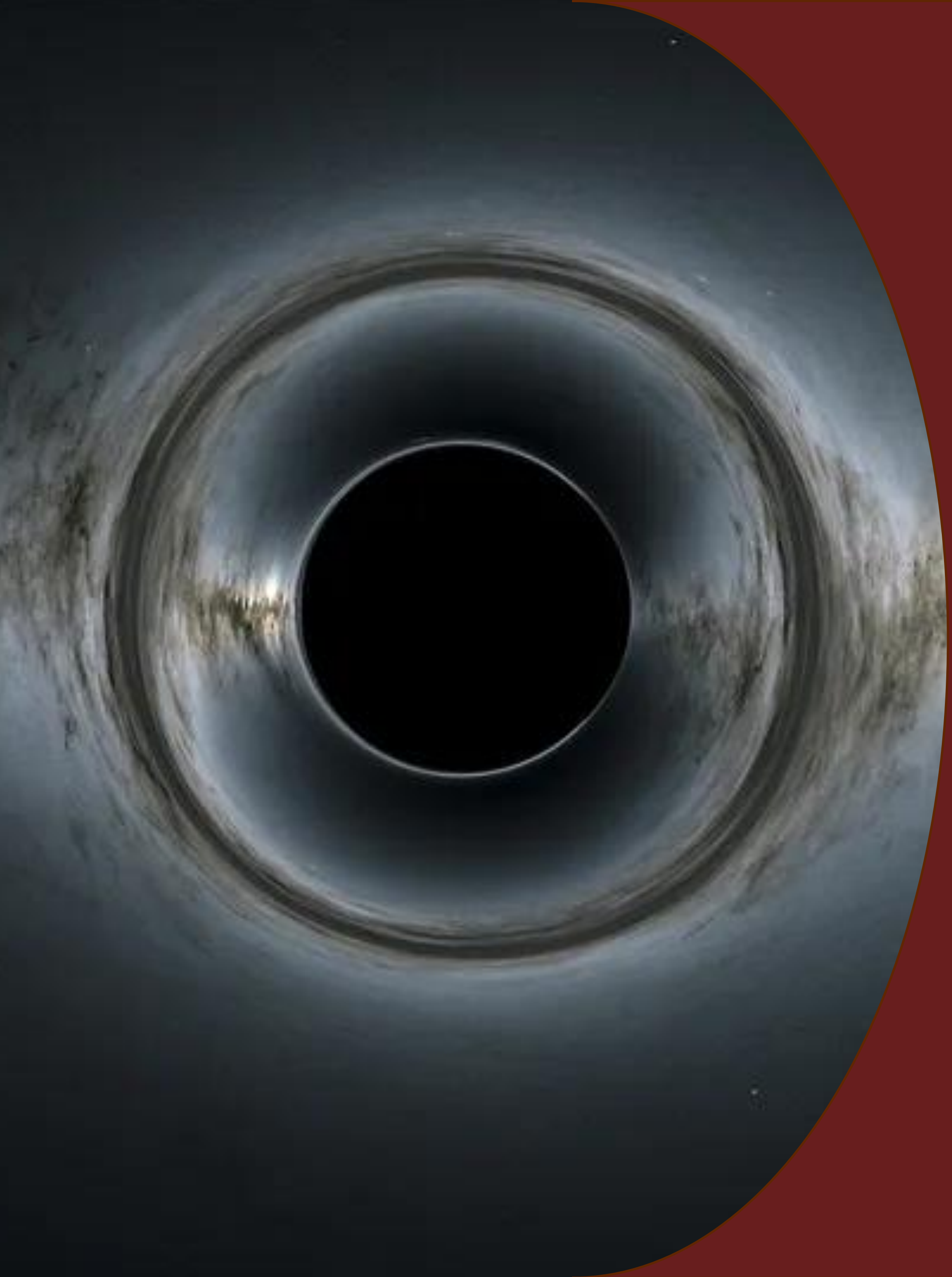

Hawking radiation in conformally static spacetimes

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Overview

- Black holes emit away particles having a thermal spectrum.
- Quantum fields in the Schwarzschild background of a dust cloud collapses to form a black hole.
- Using Bogoliubov transformation, shown that vacuum state for observer at early times evolves into an excited state with thermal particles at late times.
- Intricately connected to the second law of thermodynamics.
- Temperature determined by the surface gravity of the black hole.

Parker L, Toms D., Cambridge University Press; 2009.

Drawbacks with current models of Hawking radiation

- Outflux of particles should result in decrease of mass of black holes.

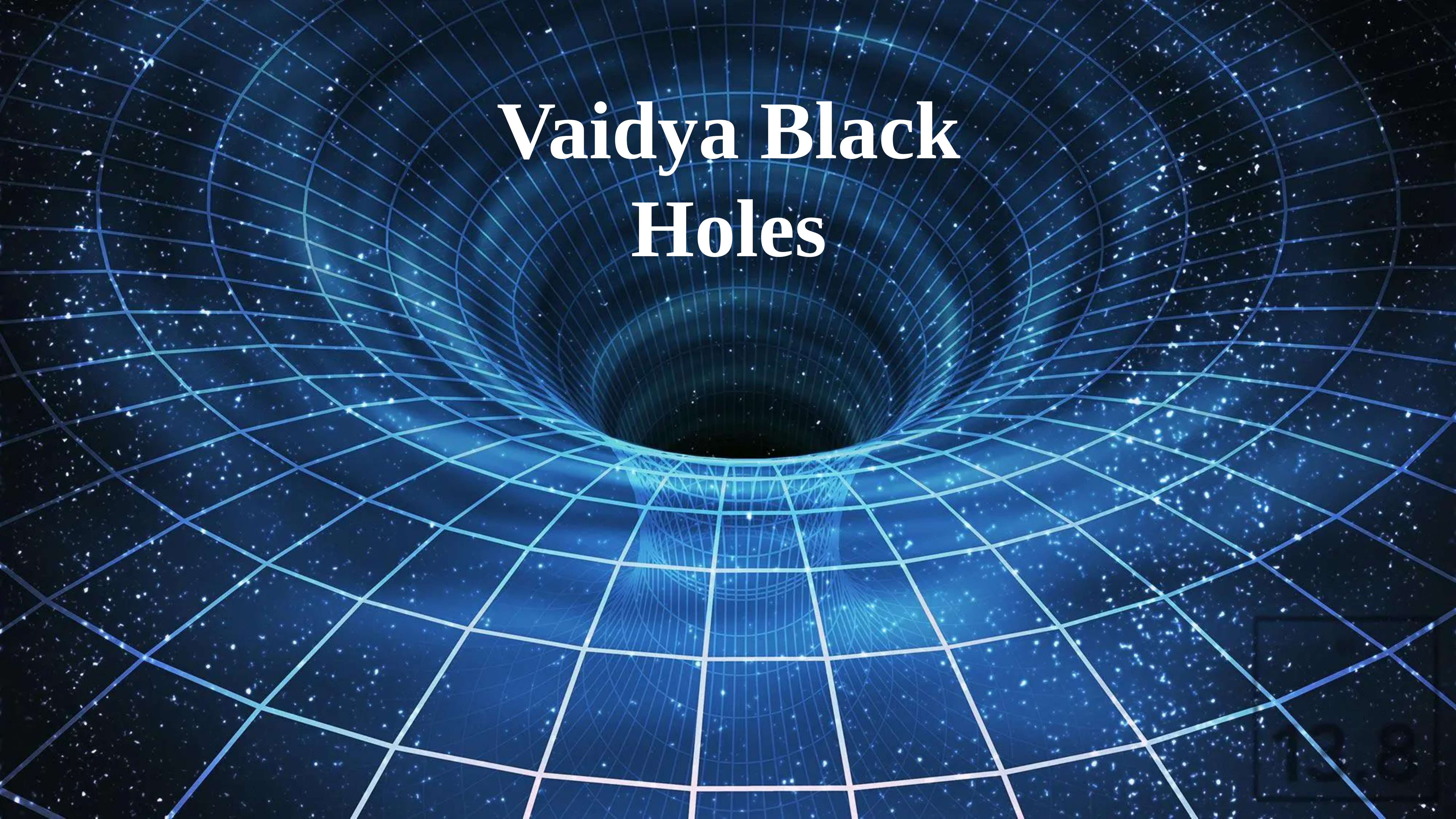
- Usual calculations don’t consider the backreaction of this outflux to the background spacetime.

- Based on eternal models of black holes.

- Generalization to dynamical spacetimes involve identifying the notion of horizon which behaves thermodynamically.

Mann, R. B., Murk, S., and Terno, D. R., 2022

Vaidya Black Holes



The Vaidya line element in advanced null coordinates is given as:

$$ds^2 = -f(v,r)\, dv^2 + 2\, dv\, dr + r^2 d\Omega^2 \;,$$

where

$$f(v,r) = 1 - \frac{r_g(v)}{r} \;.$$

The curve, $r = r_g(v)$, corresponds to the apparent horizon.

Describes radially inward moving null fluid.

Apparent horizon is a timelike surface.

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Considering a linear evaporation of the black hole, $r_g(v) = r_0 - \alpha v$ and employing the coordinate transformation,

$$v = \frac{r_0}{\alpha}\left(1 - e^{-\alpha\mathcal{T}/r_0}\right), r = Re^{-\alpha\mathcal{T}/r_0},$$
$$dT = d\mathcal{T} - \frac{1}{\left(1 - \frac{r_0}{R} + \frac{2\alpha R}{r_0}\right)} dR,$$

the line element can be expressed in a manifestly conformally static form

$$\begin{aligned} ds^2 &= e^{-2\alpha\mathcal{T}/r_0}\left[-\left(1 - \frac{r_0}{R} + \frac{2\alpha R}{r_0}\right) d\mathcal{T}^2 + 2\, d\mathcal{T}\, dR + R^2 d\Omega_2\right] \\ &= e^{-2\alpha\mathcal{T}/r_0}\left[-\left(1 - \frac{r_0}{R} + \frac{2\alpha R}{r_0}\right) dT^2 + \frac{1}{\left(1-\frac{r_0}{R}+\frac{2\alpha R}{r_0}\right)} dR^2 + R^2 d\Omega_2\right]. \end{aligned}$$

Dahal, P.K., Maharana, S., *Eur. Phys. J. C* **84**, 783 (2024).

The larger root of the following equation

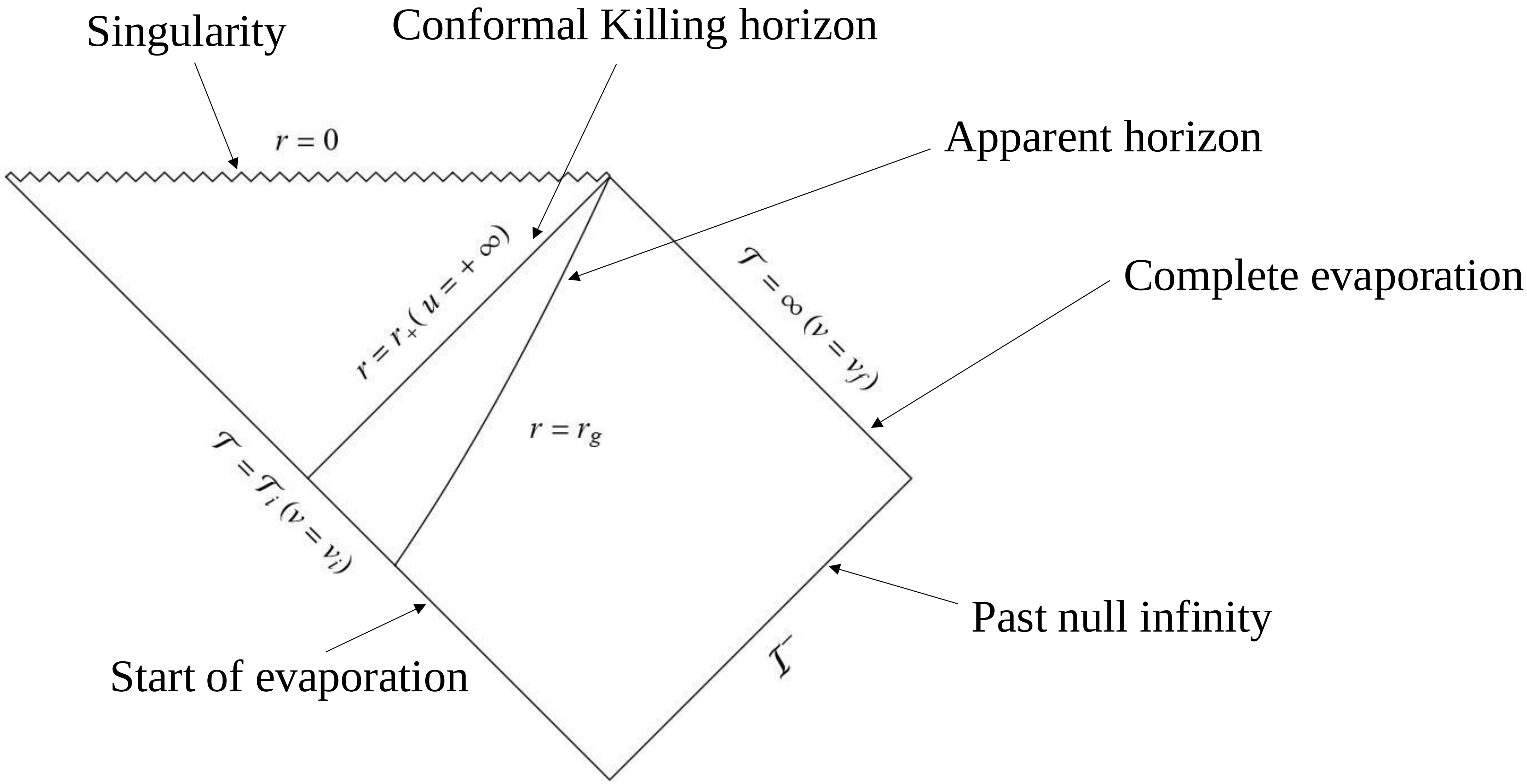
$$1-\frac{r_0}{R}+\frac{2\alpha R}{r_0}=0,$$

corresponds to the conformal Killing horizon (and even the event horizon), and satisfies the conformal Killing equation

$$\nabla_\alpha K_\beta+\nabla_\beta K_\alpha=2\,\mathcal{B}g_{\alpha\beta},$$

where

$$K^\mu=\frac{r_0}{\alpha}\left(\frac{\partial}{\partial T}\right)^\mu \text{and } \mathcal{B}=-1.$$



Singularity
Conformal Killing horizon
$r = 0$
Apparent horizon
$r = r_+ (u = + \infty)$
$\mathcal{T} = \infty \ (v = v_f)$
Complete evaporation
$r = r_g$
$\mathcal{T} = \mathcal{T}_i \ (v = v_i)$
Past null infinity
$\mathcal{I}^-$
Start of evaporation

Desired features of the evaporating Vaidya metric

- Includes dynamics.
- Candidate solution to the backreaction problem.
- Linear Vaidya metric conformal to a static metric.
- Enables to study the phenomena of particle production in dynamical spacetimes.
- May unveil aspects not seen in static background.

Mann, R. B., Murk, S., and Terno, D. R., 2022

Quantum Fields in a Linear Vaidya Background

- We consider a free massless scalar field conformally coupled to the spacetime curvature, which is characterized by the equation

$$\phi_{;\mu\nu} g^{\mu\nu} + \frac{1}{6} R^{\mu}{}_{\mu} \phi = 0.$$

- Conformally coupled field equations are invariant under conformal transformation of the form

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} \text{ and } \tilde{\phi} = \Omega^{-1} \phi.$$

where Ω is the conformal factor.

Find the solution ($\tilde{\phi}$) in the corresponding static spacetime ($\tilde{g}_{\mu\nu}$) which is conformal to the linear Vaidya spacetime ($g_{\mu\nu}$), and use conformal scaling to obtain the solution (ϕ) for the original field equation, i.e.,

$$\phi = e^{\alpha \mathcal{T}/r_0}\tilde{\phi}.$$

The most general solution of the wave equation is given by

$$\tilde{\phi} = \int dk \left(A_k f_k^{in} + A_k^\dagger f_k^{in*} + B_k f_k^{out} + B_k^\dagger f_k^{out*}\right),$$

where

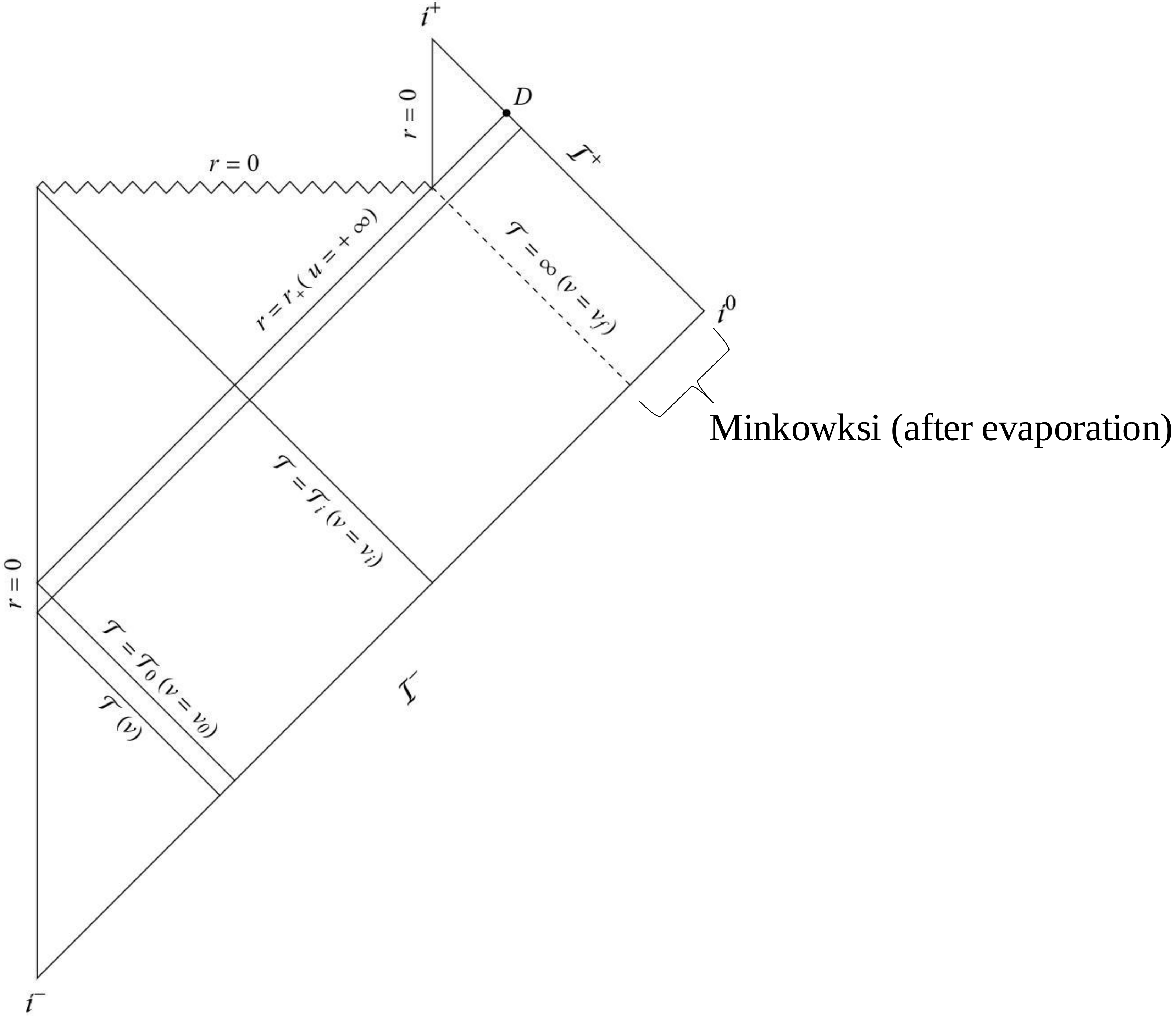
$$f_k^{in} = e^{-ik(T+R_*)}(C_1\mathcal{G}_{11}^- + C_2\mathcal{G}_{12}^-)Y_{lm}(\theta,\varphi),$$
$$f_k^{out} = e^{-ik(T-R_*)}(D_1\mathcal{G}_{11}^+ + D_2\mathcal{G}_{12}^+)Y_{lm}(\theta,\varphi).$$

Since we are interested in the massless scalar field at the past and future null infinities, $f_k^{in,\, out}\big|_{r\to\infty}$ should exist, implying

$$C_1 = 0 = D_1.$$

Tracing null rays which just escaped from the collapsed body from the asymptotic region ($\mathcal{J}^+$) backward to the past null infinity ($\mathcal{J}^-$), the affine separation is obtained to be related as follows –

$$u(\mathcal{T}) = \frac{1-\sqrt{1+8\alpha}}{2\alpha\sqrt{1+8\alpha}}\, r_0 \ln\left(\frac{\mathcal{T}_0 - \mathcal{T}}{K}\right).$$



i^+
D
$r = 0$
$r = 0$
$\mathcal{I}^+$
i^0
$r = r_+ (u = +\infty)$
$\mathcal{T} = \infty \, (v = v_f)$
Minkowksi (after evaporation)
$\mathcal{T} = \mathcal{T}_i \, (v = v_i)$
$r = 0$
$\mathcal{T} = \mathcal{T}_0 \, (v = v_0)$
$\mathcal{T} \, (v)$
$\mathcal{I}^-$
i^-

We consider incoming modes at past null infinity and outgoing modes at future null infinity

$$\phi^{in} = e^{\alpha\mathcal{T}/r_0} \int dk \left(A_k f_k^{in} + A_k^\dagger f_k^{in*} \right),$$

$$\phi^{out} = e^{\alpha\mathcal{T}/r_0} \int dk \left(B_k f_k^{out} + B_k^\dagger f_k^{out*} \right).$$

Since the outgoing modes on $\mathcal{J}^+$ are completely determined by the data on $\mathcal{J}^-$, f_k^{out} can be expressed as the linear combination of f_k^{in} and f_k^{in*}, i.e.,

$$f_k^{out} = \int dk' \left(\alpha_{kk'} f_{k'}^{in} + \beta_{kk'} f_{k'}^{in*} \right)$$

where $\alpha_{kk'} = \left(f_{k'}^{in},\ f_k^{out} \right)$ and $\beta_{kk'} = -\left(f_{k'}^{in*},\ f_k^{out} \right)$ are the Bogoliubov coefficients.

The explicit values of the Bogoliubov coefficients are obtained to be

$$\alpha_{kk'} = C \int_{-\infty}^{\mathcal{T}_0} d\mathcal{T} \left(\frac{k'}{k}\right)^{1/2} e^{-iku} e^{ik'\mathcal{T}} ,$$

$$\beta_{kk'} = C \int_{-\infty}^{\mathcal{T}_0} d\mathcal{T} \left(\frac{k'}{k}\right)^{1/2} e^{-iku} e^{-ik'\mathcal{T}} .$$

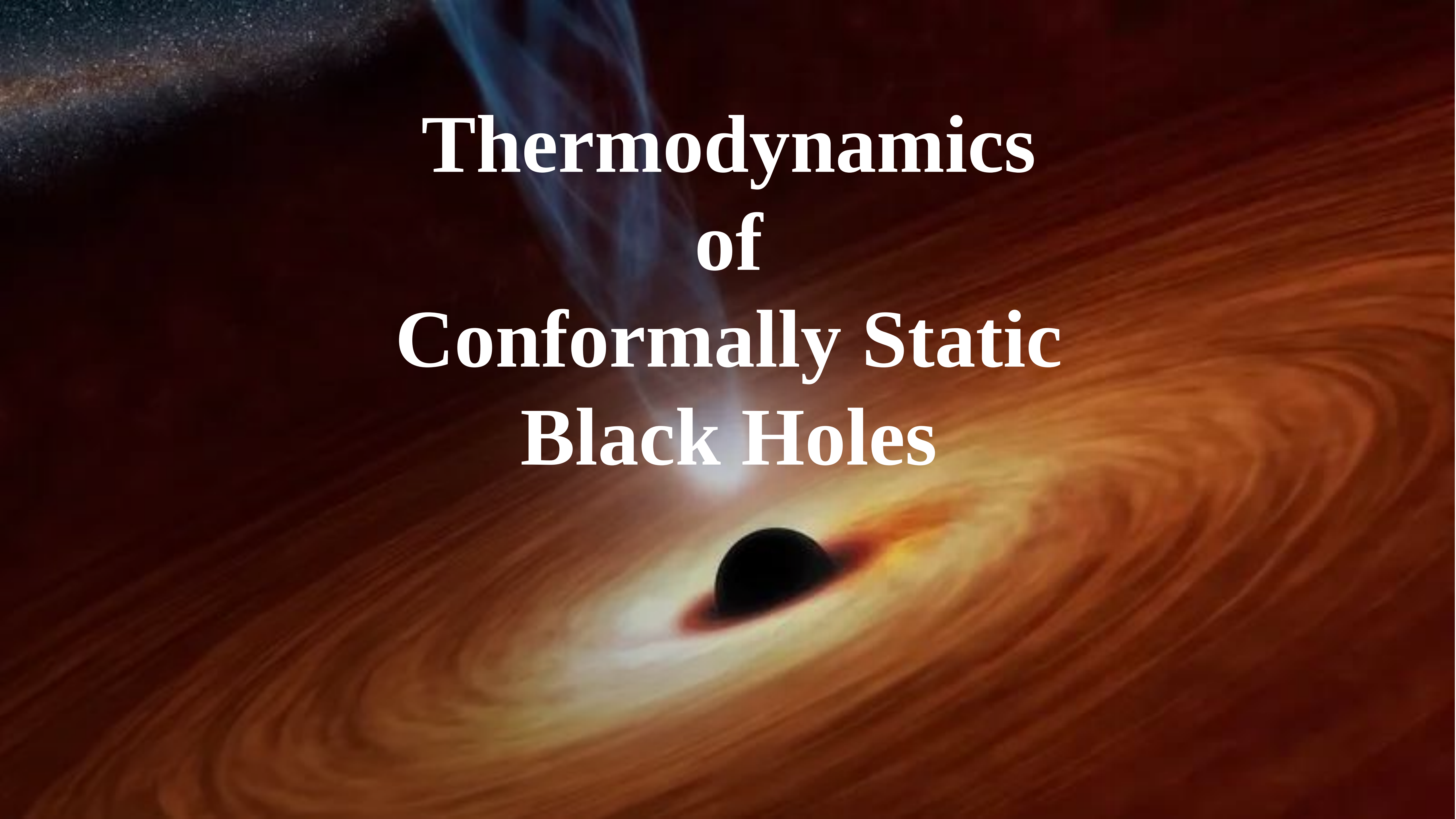
This renders the relation

$$|\alpha_{kk'}|^2 = e^{\frac{\pi k\left(\sqrt{1+8\alpha}-1\right) r_0}{\alpha\sqrt{1+8\alpha}}} |\beta_{kk'}|^2$$

Hence, Vaidya BHs exhibit emission and absorption behaviour analogous to a gray body of absorptivity $\Gamma(k)$ and temperature T

$$T = \frac{\alpha\sqrt{1+8\alpha}}{\pi\left(\sqrt{1+8\alpha}-1\right)r_0} = \frac{\kappa}{2\pi}.$$

Dahal, P.K., Maharana, S., *Eur. Phys. J. C* **84**, 783 (2024).


Thermodynamics
of
Conformally Static
Black Holes

- Laws of black hole mechanics analogous to the laws of thermodynamics.
- There exist an equivalence between black hole macroscopic parameters with the thermodynamic parameters.
- Enables a statistical treatment of black holes which can give information about the black hole internal structure.
- Based on the existence of a horizon.

Smarr’s formula relates the black hole mass (M), the angular momentum (J) and the black hole area (A)

$$M = 2\Omega_H J + \frac{\kappa A}{4\pi}.$$

Parametric perturbation of the black hole parameters to $M + \delta M$ and $J + \delta J$ renders *the first law of black hole mechanics*

$$\delta M = \frac{\kappa\, \delta A}{4\pi} + \Omega_H\, \delta J.$$

E. Poisson, *A Relativist’s Toolkit*, Cambridge University Press (2004)

Since we consider the conformal Killing horizon (or the event horizon, in this case) to be behaving thermodynamically, we expect it to be show up in the first law of black hole mechanics.

The conformal Killing vector, K^μ, satisfies the conformal Killing equation

$$K_{\mu;\nu} + K_{\nu;\mu} = 2\mathcal{B} g_{\mu\nu}.$$

The conformal Killing horizon is the null hypersurface characterised by the condition

$$K^\mu K_\mu = 0$$

We introduce the conformally invariant definition of surface gravity on the conformal Killing horizon as follows –

$$\left(K^{\mu}K_{\mu}\right)_{;\nu} = -2\kappa K_{\nu}$$

which can be equivalently cast in the form

$$\kappa = \left(K_{\mu;\nu} - 2\mathcal{B}g_{\mu\nu}\right)N^{\mu}\hat{T}^{\nu},$$

κ is constant on the conformal Killing horizon

Equivalent to zeroth law of black hole mechanics

Proceeding in the same way as deriving the Komar mass, we integrate the following equation

$$K^{\mu}{}_{;\beta\mu} = R_{\nu\beta}K^{\nu} + \left(\mathcal{B}g^{\mu}{}_{\mu}\right)_{,\beta}$$

Over some spacelike hypersurface Σ, which renders

$$\int_{\Sigma} K^{\mu}{}_{;\beta\mu}\hat{T}^{\beta}\sqrt{\left|\gamma^{(\Sigma)}\right|}d^{3}x = \int_{\Sigma} \left(R_{\nu\beta}K^{\nu} + \left(\mathcal{B}g^{\mu}{}_{\mu}\right)_{,\beta}\right)\hat{T}^{\beta}\sqrt{\left|\gamma^{(\Sigma)}\right|}d^{3}x$$

Where $\gamma^{(\Sigma)}$ is the induced metric on Σ and $\hat{T}^{\beta}$ is a unit timelike vector normal to Σ.

Since the quantity $\left(K_{\mu;\nu} - \mathcal{B} g_{\mu\nu}\right)$ is anti-symmetric, we can readily convert the volume integral on the left hand side to a surface integral, i.e.,

$$\int_{\partial\Sigma_\infty} \left(K_{\mu;\nu} - \mathcal{B} g_{\mu\nu}\right) N^\mu \hat{T}^\nu \sqrt{\left|\gamma^{(\partial\Sigma_\infty)}\right|} d^2y$$
$$- \int_{\partial\Sigma_+} \left(K_{\mu;\nu} - \mathcal{B} g_{\mu\nu}\right) N^\mu \hat{T}^\nu \sqrt{\left|\gamma^{(\partial\Sigma_+)}\right|} d^2y$$
$$= \int_\Sigma \left(R_{\nu\beta} K^\nu + \left(\mathcal{B} g^\mu{}_\mu\right)_{,\beta}\right) \hat{T}^\beta \sqrt{\left|\gamma^{(\Sigma)}\right|} d^3x\,,$$

where $\partial\Sigma_\infty$ and $\partial\Sigma_+$ bound the volume Σ.

$$M = \frac{1}{4\pi}\int_{\Sigma} \left(R_{\nu\beta}K^{\nu} + \left(\mathcal{B}g^{\mu}{}_{\mu}\right)_{,\beta}\right)\hat{T}^{\beta}\sqrt{\left|\gamma^{(\Sigma)}\right|}d^3x + \frac{\kappa}{4\pi}A - \frac{1}{4\pi}\int_{\partial\Sigma_+} \mathcal{B}\, dA\,.$$

For slowly evaporating BHs (such as macroscopic BHs), $\mathcal{B}$ can be taken to be constant so that

$$M = \frac{1}{4\pi}\int_{\Sigma} \left(R_{\nu\beta}K^{\nu} + \left(\mathcal{B}g^{\mu}{}_{\mu}\right)_{,\beta}\right)\hat{T}^{\beta}\sqrt{\left|\gamma^{(\Sigma)}\right|}d^3x + \frac{\kappa - \mathcal{B}}{4\pi}A\,.$$

Can read off the effective temperature to be $\frac{\kappa-\mathcal{B}}{4\pi}$, that contains contributions from both Hawking radiation and classical black hole emission.

The first law can be obtained by considering the changes due to parametric differences between infinitesimal diffeomorphic solutions $g'_{\mu\nu} = g_{\mu\nu} + \delta g_{\mu\nu}$

$$\delta M = \frac{1}{8\pi}\int_{\partial\Sigma_+} \big(9k_0 - 2(k_3 + k_6)\big)\mathcal{B}\, dA + \frac{1}{8\pi}\int_{\partial\Sigma_+} (\kappa - \mathcal{B})\delta dA + \text{matter terms}$$

In analogy with the static case, the second term gives the effective temperature.

However, no equivalent of the first term in the static case.

Attributed to the mass change brought about by the black hole dynamics.

Dahal, P.K., Maharana, S., *Eur. Phys. J. C* **84**, 783 (2024).

Takeaways

- $\mathcal{B} = \text{const}$ for the linearly evaporating case, rendering the effective temperature constant.

- For conformal spacetimes, there exist static coordinates (T, R).

- Allows us to formulate laws of thermodynamics similar to those of static black holes without relying on adiabaticity condition.

- Event horizon behaves thermodynamically.

- Not applicable for horizonless compact objects.

Future Works

- An equivalent formulation of the second law and third law of black hole mechanics for conformally static spacetimes is imminent.

- Introduce rotation to the current scheme and look at its effect.

- Look at particle production behaviour of a general spherically symmetric conformally static spacetime.

- The existence of the conformal Killing horizon within the apparent horizon facilitates a Penrose process within the quantum ergosphere region.

THANK YOU !!!