Gravitational waves: a numerical exploration of the global scattering problem

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ACGRG, September 2024

Outline of talk

- The global scattering problem
- New non-linear results
- New linear results for initial data
- Future perspectives

- New linear results for evolutions – ask me afterward!

Gravitational waves are not locally well defined

- No background geometry as a reference. The wave *is* the geometry!

Figure: Image by brgfx on Freepik

To infinity!

- For asymptotically flat space-times, a well defined prescription exists at *infinity*.
- These space-times model isolated systems, and their curvature falls off to zero at infinity.
- This prescription can also be thought of as idealizing *local* behaviour.

Asymptotic simplicity

Definition

A smooth (time- and space-orientable) Lorentzian space-time $(\widetilde{M}, \widetilde{g}_{ab})$ is called asymptotically simple, if there exists another smooth Lorentzian space-time (M, g_{ab}) such that

- $\widetilde{M}$ is an open submanifold of M with smooth boundary $\partial\widetilde{M} = \mathscr{I}$;
- there exists a smooth scalar field Θ on M, such that $g_{ab} = \Theta^2 \widetilde{g}_{ab}$ on $\widetilde{M}$, and so that $\Theta = 0$, $d\Theta \neq 0$ on $\mathscr{I}$;
- every null geodesic in $\widetilde{M}$ acquires a future and a past endpoint on $\mathscr{I}$.

An asymptotically simple space-time is called asymptotically flat, if in addition $\widetilde{R}_{ab} = 0$ in a neighbourhood of $\mathscr{I}$.

Minkowski space-time $(\tilde{M}, \tilde{g})$ embedded in the Einstein cylinder $(M, \Theta^2 \tilde{g})$

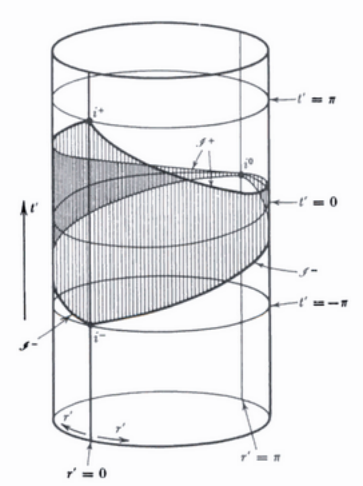
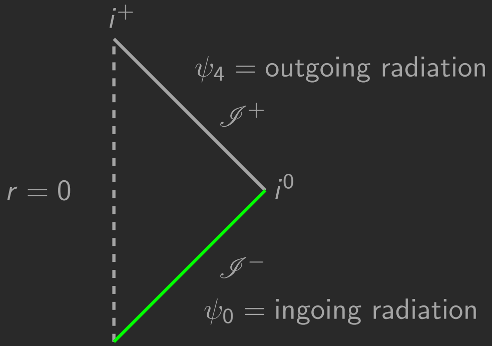


Figure: From *The large scale structure of space-time* Hawking and Ellis

The global scattering problem



Set the non-interacting initial data on $\mathscr{I}^-$ (in-states) and relate to the non-interacting final data on I^+ (out-states).

Work done so far

- Analytical:
  - Conformal scattering (Nicolas, Mason, Friedlander, ...).
- Numerical:
  - Fully psuedo spectral methods for conformally invariant wave equation on Minkowski, Schwarzschild or Kerr space-times (Hennig, Frauendiener, Macedo)
  - Evolution of linearised spin-2 equation around Minkowski space-time (Frauendiener, Doulis)
  - No non-linear Einstein equations, no linear Einstein equations from $\mathscr{I}^-$ to $\mathscr{I}^+$ has been done.

Conformal Field Equations

- H. Friedrich devised a regular extension of the Einstein equations to the conformal space-time, The Conformal Field Equations.

$$e_{\mathbf{a}}(c^{\mu}_{\mathbf{b}}) - e_{\mathbf{b}}(c^{\mu}_{\mathbf{a}}) = \widehat{\Gamma}_{\mathbf{ab}}{}^{\mathbf{c}} c^{\mu}_{\mathbf{c}} - \widehat{\Gamma}_{\mathbf{ba}}{}^{\mathbf{c}} c^{\mu}_{\mathbf{c}},$$

$$e_{\mathbf{a}}(\widehat{\Gamma}_{\mathbf{bc}}{}^{\mathbf{d}}) - e_{\mathbf{b}}(\widehat{\Gamma}_{\mathbf{ac}}{}^{\mathbf{d}}) = \left(\widehat{\Gamma}_{\mathbf{ab}}{}^{\mathbf{e}} - \widehat{\Gamma}_{\mathbf{ba}}{}^{\mathbf{e}}\right) \widehat{\Gamma}_{\mathbf{ec}}{}^{\mathbf{d}} - \widehat{\Gamma}_{\mathbf{bc}}{}^{\mathbf{e}} \widehat{\Gamma}_{\mathbf{ae}}{}^{\mathbf{d}} + \widehat{\Gamma}_{\mathbf{ac}}{}^{\mathbf{e}} \widehat{\Gamma}_{\mathbf{be}}{}^{\mathbf{d}}$$
$$+ \Theta K_{\mathbf{abc}}{}^{\mathbf{d}} - 2\eta_{\mathbf{c}[\mathbf{a}} \widehat{P}_{\mathbf{b}]}{}^{\mathbf{d}} + 2\delta_{[\mathbf{a}}{}^{\mathbf{d}} \widehat{P}_{\mathbf{b}]\mathbf{c}} - 2\widehat{P}_{[\mathbf{ab}]} \delta_{\mathbf{c}}{}^{\mathbf{d}},$$

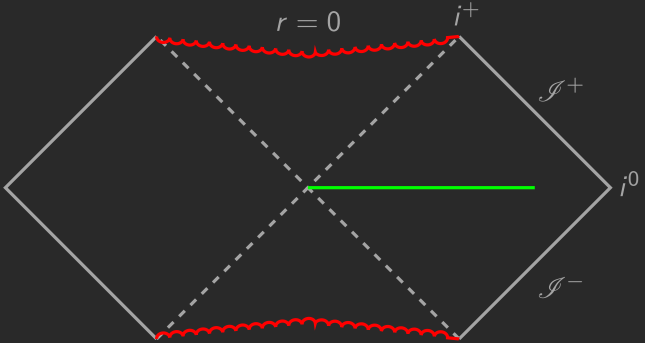
$$\widehat{\nabla}_a \widehat{P}_{bc} - \widehat{\nabla}_b \widehat{P}_{ac} = h_e K_{abc}{}^{e},$$

$$\nabla_e K_{abc}{}^{e} = 0, \qquad [K_{abc}{}^{d} = \Theta^{-1} C_{abc}{}^{d} = \Theta^{-1} \tilde{C}_{abc}{}^{d}]$$

- This is a first order, non-linear, symmetric hyperbolic system for the frame components, connection and curvature.

Success of the numerical IBVP framework

Papers [1] [2] [3]



[1] Frauendiener, J., & Stevens, C. (2021). The non-linear perturbation of a black hole by gravitational waves. I. The Bondi–Sachs mass loss. Classical and Quantum Gravity, 38(19), 194002.

[2] Frauendiener, J. & Stevens, C. (2023). The non-linear perturbation of a black hole by gravitational waves. II. Quasinormal modes and the compactification problem. Classical and Quantum Gravity, 40(12), 125006.

[3] Goodenbour, A, Frauendiener, J. & Stevens, C. (2024). The non-linear perturbation of a black hole by gravitational waves. III. Newman-Penrose constants. Classical and Quantum Gravity 41(6), 065005.

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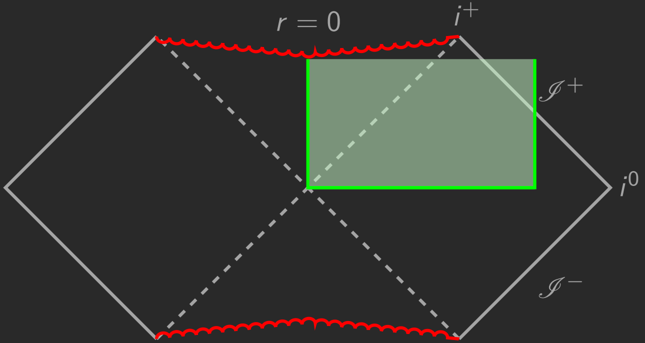

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Fully non-linear results for scattering with a black hole

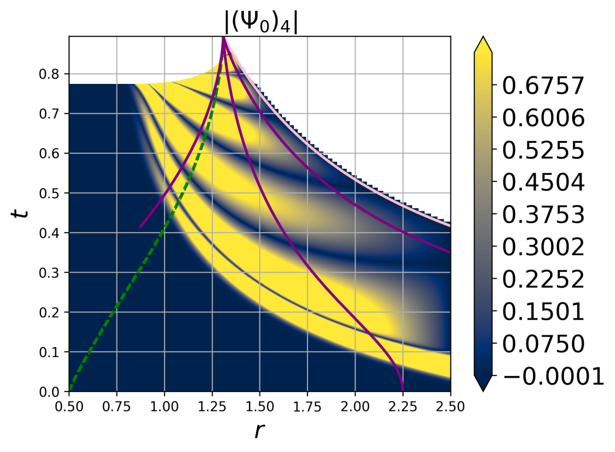
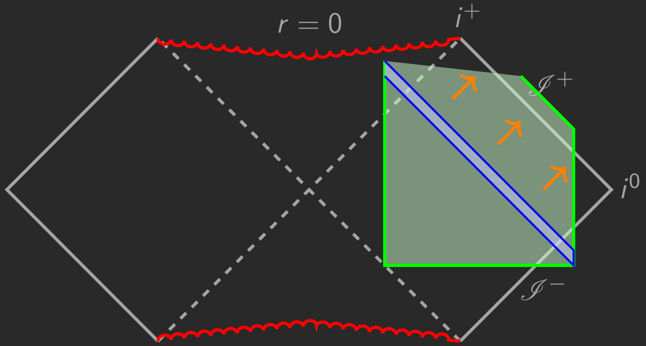
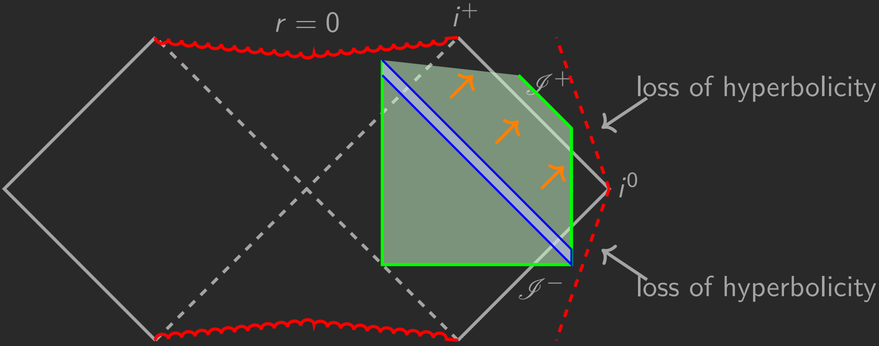


Figure: From Frauendiener, J. & Stevens, C. (2023). The non-linear perturbation of a black hole by gravitational waves. II. Quasinormal modes and the compactification problem. Classical and Quantum Gravity, 40(12), 125006.

Step toward global non-linear scattering problem



Step toward global non-linear scattering problem



Convergence test on $\mathscr{I}^-$

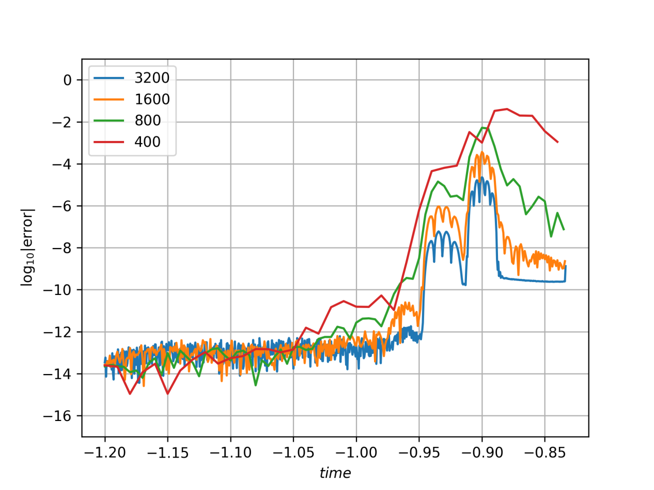


Figure: A constraint from $\nabla^a K_{abcd} = 0$ along $\mathscr{I}^-$

Convergence test on $\mathscr{I}^+$

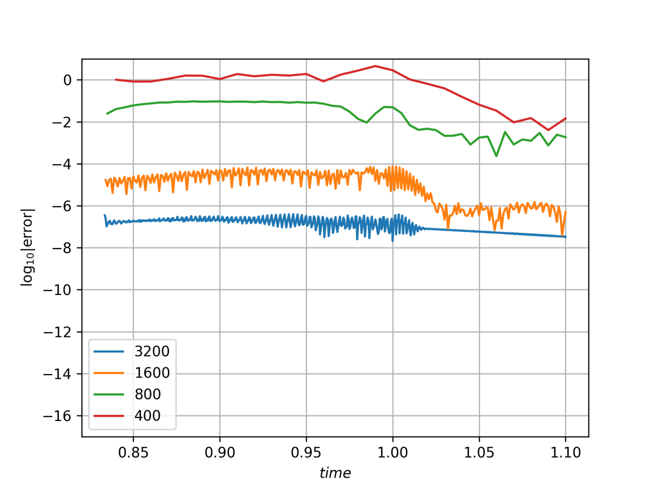
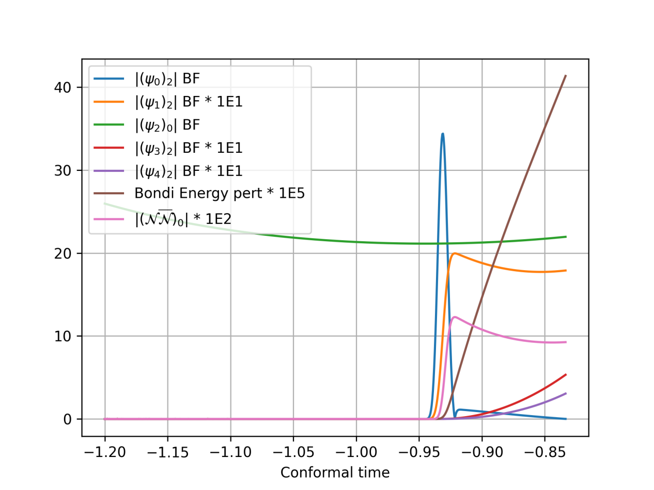


Figure: A constraint from $\nabla^a K_{abcd} = 0$ along $\mathscr{I}^+$

Bondi energy and gravitational data on $\mathscr{I}^-$



Bondi energy and gravitational data on $\mathscr{I}^+$

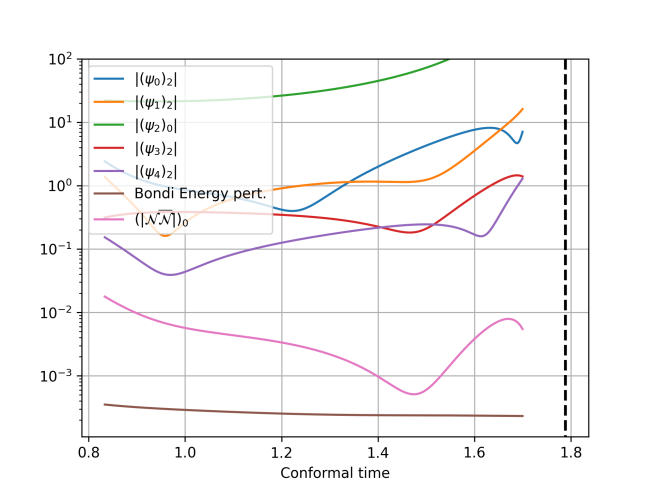
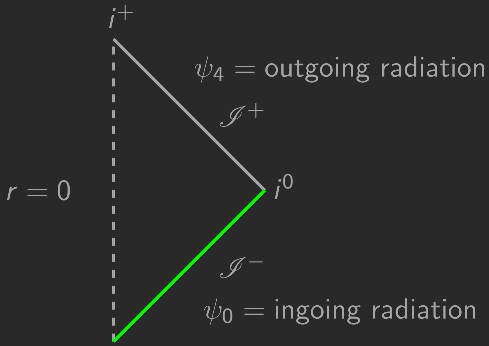


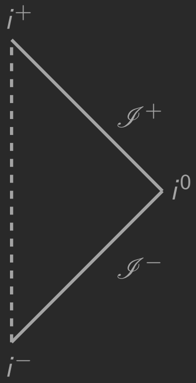
Figure: The vertical dashed line corresponds to future timelike infinity

The global scattering problem – linearised

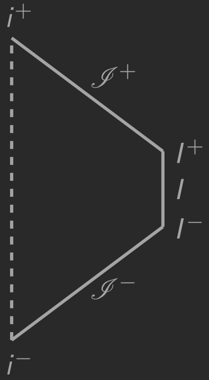


Set the non-interacting initial data on $\mathscr{I}^-$ (in-states) and relate to the non-interacting final data on I^+ (out-states).

The cylinder at spatial infinity



(a) There are pathologies at spatial infinity (i^0).



(b) These are easier to deal with if we blow up spatial infinity from a point to a cylinder

The linearised equations on $\mathscr{I}^-$

- Restricting the system to $\mathscr{I}^-$, we see that 4 of the equations are intrinsic to $\mathscr{I}^-$.

- Using a 'fully-compactified' gauge that covers the whole space-time where $r = 0$ is past timelike infinity and $r = 1$ is the bottom of the cylinder, and expanding in SWSH, the 4 equations on $\mathscr{I}^-$ are

$$\frac{\partial \psi_k}{\partial r} = -\frac{\pi(3 + (7 - 2k)c(2r))}{2s(2r)}\psi_k + \frac{\pi a_{k-1}}{s(2r)}\psi_{k-1},$$

where $k = 1, 2, 3, 4,\ a_k = \sqrt{l(l+1) - (k-1)(k-2)}$,

$c(r) = \cos\left(\frac{\pi r}{2}\right)$ and $s(r) = \sin\left(\frac{\pi r}{2}\right)$.

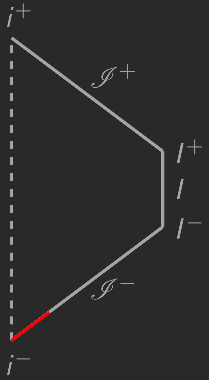
- Integrating these yields

$$\psi_k(r) = \frac{\pi a_k}{2} s(r)^{k-5} c(r)^{k-2} \left(\int_0^r s(\rho)^{4-k} c(\rho)^{1-k} \psi_{k-1} \mathrm{d}\rho \right).$$

Regularity conditions

- There have existed regularity conditions in the neighbourhood of i^- and I^- separately, but not together:

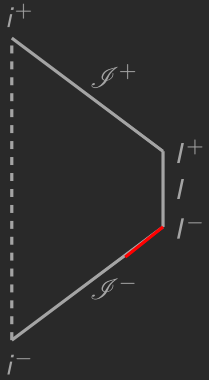
$$\text{If } \psi_0 = O(r^n) \text{ then } \psi_k = O(r^n) \text{ also.}$$



Regularity conditions

- There have existed regularity conditions in the neighbourhood of i^- and I^- separately, but not together:

$$\text{If } \psi_0 = O((r-1)^n) \text{ then } \psi_k = O((r-1)^n) \text{ also.}$$



Regularity conditions for all of $\mathscr{I}^-$

Theorem (Barrer, Stevens)

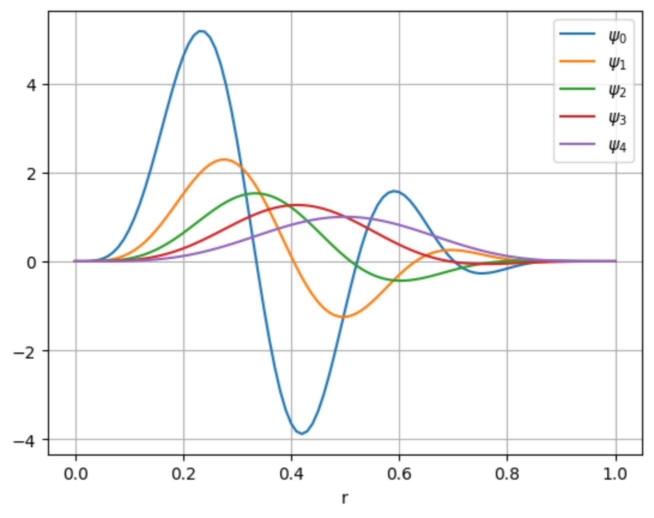
If ψ_0 has compact support $(a, b) \subseteq (0, 1)$ with fall-off $\mathcal{O}((1-r)^3)$ or greater towards $r = 1$ (the bottom of the cylinder I^-), and $\psi_k(r) = 0$ for $r \leq a$, then ψ_k is finite at $r = 0$ for all k if and only if

$$\int_a^b \sin\left(\frac{\pi r}{2}\right)^3 \psi_0(r)\,\mathrm{d}r = 0. \tag{1}$$

Additionally, $\psi_k(1) = 0$ for all k if and only if

$$\int_a^b \cot(\pi r)\sin\left(\frac{\pi r}{2}\right)^3 \psi_0(r)\,\mathrm{d}r = 0. \tag{2}$$

Solving for the initial data



Next 30 years?

- A numerical framework that can calculate a large class of initial data on $\mathscr{I}^-$ and then use it to calculate the entire resulting asymptotically-flat space-time. This can then be used to, in a reasonably unambiguous way, simulate local pieces of our universe.

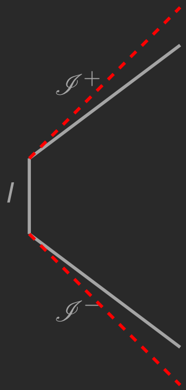
Evolving the linearized Equations off of $\mathscr{I}^-$

In a semi-compactified gauge around the cylinder at spatial infinity, we have

$$
\begin{aligned}
(1+t\kappa')\partial_t\psi_0 &= \kappa\partial_r\psi_0 - (3\kappa'-\mu)\psi_0 - \mu\alpha_2\psi_1,\\
\partial_t\psi_1 &= -\mu\psi_1 + \frac{1}{2}\mu\alpha_2\psi_0 - \frac{1}{2}\mu\alpha_0\psi_2,\\
\partial_t\psi_2 &= \frac{1}{2}\mu\alpha_0\psi_1 - \frac{1}{2}\mu\alpha_0\psi_3,\\
\partial_t\psi_3 &= \mu\psi_3 + \frac{1}{2}\mu\alpha_0\psi_2 - \frac{1}{2}\mu\alpha_2\psi_4,\\
(1-t\kappa')\partial_t\psi_4 &= -\kappa\partial_r\psi_4 + (3\kappa'-\mu)\psi_4 + \mu\alpha_2\psi_3
\end{aligned}
$$

These equations are symmetric hyperbolic except for at the *Critical Sets*, $t=\pm\frac{1}{\kappa'}$. These intersect the top and bottom of the cylinder.

Loss of Hyperbolicity



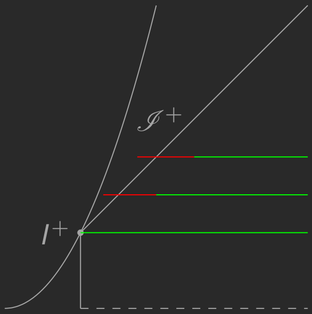
Evolving through I^-

Relatively simple, just use an implicit method!

$$y_{n+1} = y_n + dt * f(y_{n+1}, t + dt) \tag{3}$$

This avoids the evaluation at I^-, problem solved. However I^+ is more difficult.

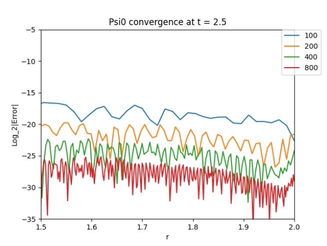
Evolving through I^+



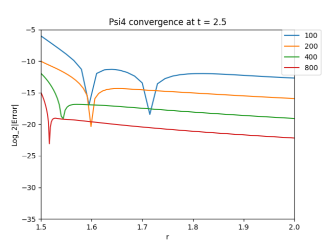
Evolving through I^+ continued

$$\partial_r \psi_4 = \frac{1}{r(1+r)}\left(((1+r)^2 - t)\partial_t \psi_4 + (2-r)\psi_4 - 2(1+r)\psi_4\right),$$

Checks of correctness above the cylinder

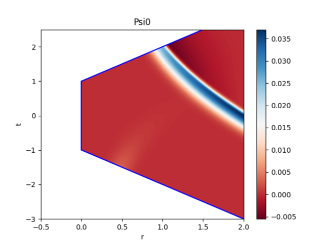


(a) $\log_2$Error of ϕ_0

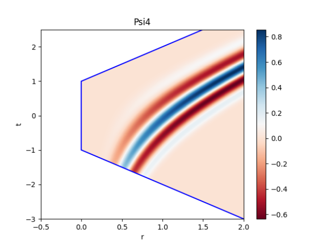


(b) $\log_2$Error of ϕ_4

Contour plots reproducing another solution



(a) ϕ_0



(b) ϕ_4